Loop Quantum Gravity and Quantum Information

圈量子引力与量子信息

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Contents

目录

Abstract

摘要

We summarize recent developments at the interface of quantum gravity and quantum information and discuss applications to the quantum geometry of space in loop quantum gravity. In particular, we describe the notions of link entanglement, intertwiner entanglement, and boundary spin entanglement in a spin-network state. We discuss how these notions encode the gluing of quanta of space and their relevance for the reconstruction of a quantum geometry from a network of entanglement structures. We then focus on the geometric entanglement entropy of spin-network states at fixed spins, treated as a many-body system of quantum polyhedra, and discuss the hierarchy of volume-law, area-law, and zero-law states. Using information theoretic bounds on the uncertainty of geometric observables and on their correlations, we identify area-law states as the corner of the Hilbert space that encodes a semiclassical geometry and the geometric entanglement entropy as a probe of semiclassicality.

我们总结量子引力与量子信息交叉领域的最新进展，讨论其在圈量子引力空间量子几何中的应用。我们具体介绍了自旋网络态中的链接纠缠、intertwinner 纠缠和边界自旋纠缠概念，阐述这些概念如何编码空间量子的粘接，以及它们对从纠缠结构网络重构量子几何的重要性。随后我们聚焦固定自旋下自旋网络态的几何纠缠熵——该态被视为量子多面体多体系统，讨论了体积律、面积律和零律法态的层级结构。利用信息论对几何观测量的不确定性及其关联给出的界，我们确定面积律态是希尔伯特空间中编码半经典几何的区域，而几何纠缠熵是半经典性的探针。

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Keywords

关键词

Loop quantum gravity - Spin networks - Quantum information . Entanglement - Area-entropy law

圈量子引力-自旋网络-量子信息。纠缠-面积熵定律

Introduction

引言

A physical and predictive description of gravitation at all scales of length and energy, or in short quantum gravity, is meant to emanate from consistently merging quantum theory and general relativity. In quantizing gravity or gravitizing the quantum, whichever perspective or starting point one might choose, a common thread is that we need to push further the revolutionary intuition of general relativity: gravity is not defined as a mere force or interaction but arises from the relations between observers; it is a manifestation of the relativity principle itself, applied to all observers and not restricted to inertial observers.

对所有长度与能量尺度下的引力给出兼具物理意义与可预言性的描述——也就是简称为量子引力的研究方向，目标是从量子理论与广义相对论的自洽融合中诞生。无论我们选择哪种视角或出发点，是量子化引力还是让量子适配引力，一条共通的思路是，我们需要进一步推进广义相对论的革命性洞见: 引力绝非单纯的力或相互作用，它起源于观测者之间的关联；它是相对性原理本身的体现，适用于所有观测者，而非仅局限于惯性观测者。

With geometry weaved from the network of relations between space-time points, it is then natural to seek a reformulation of gravity entirely in terms of information and flow. This point of view becomes even more relevant at the full quantum level. When the classical notions of coordinate system and reference frame fade away, the fundamental symmetry of general relativity - diffeomorphism invariance - ineluctably leads to a localization problem. With no direct way to localize quantum systems in a quantum space-time without background geometry, only relations are physical degrees of freedom (see, for instance, [1] for such a reformulation of quantum mechanics). In this context, quantum geometry should be entirely defined in algebraic terms without referring to a classical background. Translating it to the language of quantum information would allow us to better understand its structure, in particular the flow from microscopic scales to macroscopic structures and the emergence of a classical smooth geometry.

> 既然几何由时空点之间的关联网络编织而成，自然就需要我们完全从信息与流动的角度对引力重新表述。这一观点在完整量子层面的重要性更加凸显。当坐标系与参考系的经典概念逐渐失效，广义相对论的基本对称性——微分同胚不变性——不可避免地引出了局域化问题。在没有背景几何的量子时空中，无法直接对量子系统定位，只有关联才是物理自由度 (例如，[1] 中就给出了量子力学的此类表述)。在此背景下，量子几何应当完全用代数语言定义，无需引入经典背景。将其转换为量子信息语言能帮助我们更好地理解它的结构，尤其是从微观尺度到宏观结构的流动，以及经典光滑几何的涌现过程。

Among the several approaches to quantum gravity, loop quantum gravity (LQG) builds space-time from Planck scale quanta of geometry. The fundamental degree of freedom is not the metric, but the Ashtekar-Barbero connection encoding the transport of reference frames along the space-time manifold. The metric is a composite field, distances are an emergent concept, and the quantum space-time is built like an evolving lego system with quantum superpositions and quantum evolution of quantized reference frames.

> 在多种量子引力研究路径中，圈量子引力 (LQG) 从普朗克尺度的几何量子出发构建时空。基本自由度不是度规，而是编码参考系沿时空流形输运的阿希提卡-巴贝罗联络。度规是复合场，距离是涌现概念，量子时空就像一套演化的乐高系统，包含量子化参考系的量子叠加与量子演化。

In this context, we would like to use quantum information, and more specifically quantum entanglement, as a witness of quantum geometry, in order to probe the geometry at different scales and understand how classical geometry emerges. In fact, a starting point is the natural hierarchy of correlations. For instance, two-point correlation functions, in quantum field theories and condensed matter models, naturally allow to reconstruct distances between points, in suitable regimes. Then we expect three-point correlations reveal corrections to the triangular inequality for distances and mismatch of transport around loops, i.e., to the notion of curvature and, so on, with multi-body entanglement revealing further layers of information about the geometry.

> 在此背景下，我们希望借助量子信息，更具体来说是量子纠缠，作为量子几何的观测工具，来探测不同尺度的几何，理解经典几何如何涌现。实际上，我们的出发点是关联的自然层级结构。例如，量子场论和凝聚态模型中的两点关联函数，可以在合适的区间内自然重建点与点之间的距离；我们预期三点关联能揭示距离三角不等式的修正，以及绕闭回路输运的不匹配——也就是曲率概念的修正，以此类推，多体纠缠会揭示几何更深层的信息。

This chapter is thus devoted to introducing the basic mathematical notions to understand and study the structure of entanglement for spin network states of geometry in loop quantum gravity and to providing useful examples illustrating the various sources of entanglement for quantum geometries.

> 因此，本章旨在介绍理解与研究圈量子引力中几何自旋网络态纠缠结构的基础数学概念，并给出示例说明量子几何不同来源的纠缠。

Quantum States of Geometry

几何量子态

Loop quantum gravity is based on a Hamiltonian formulation of general relativity. At the classical level, we proceed to a 3+1 decomposition of space-time into a three-dimensional space and the one-dimensional time. Then general relativity predicts how the three-dimensional space manifold evolves in time. At the quantum level, loop quantum gravity defines quantum states of 3D geometry and describes their evolution in time.

> 圈量子引力建立在广义相对论的哈密顿表述之上。经典层面上，我们将时空 3+1 分解为三维空间和一维时间，广义相对论由此给出三维空间流形的时间演化规律。量子层面上，圈量子引力定义了三维几何的量子态，并描述了它们的时间演化。

The quantum geometry is not directly described in terms of metric. As in other approaches to quantum gravity, the metric is a semiclassical notion emerging at large scales, in suitable regimes, from more fundamental quantum degrees of freedom. Loop quantum gravity (LQG) relies on the first-order formulation of GR in terms of tetrad and Lorentz connection. The tetrad defines the local choice of basis for the four space-time dimensions, while the connection describes how to transport this local basis from a space-time point to another. The metric is a composite field reconstructed from both tetrad and connection. In those variables, classical general relativity is written as a gauge theory. The theory is invariant under both space-time diffeomorphisms and the Lorentz group as the local gauge group. We expect those symmetries to also drive the quantum theory.

> 量子几何并非直接用度规描述。和其他量子引力方案一样，度规是半经典概念，由更基础的量子自由度在大尺度合适条件下演生而来。圈量子引力 (LQG) 采用广义相对论的一阶表述，以标架和洛伦兹联络作为基本变量。标架定义了时空四个维度的局域基矢选择，联络则描述如何将这个局域基矢从一个时空点平移到另一个时空点。度规是由标架和联络共同重构得到的复合场。在这些变量下，经典广义相对论可以写为规范理论，该理论在时空微分同胚和局域规范群洛伦兹群下保持不变，我们预期这些对称性也会支配量子理论。

LQG focuses on a complete set of observables: the holonomies (encoding the finite transport between space points defined by the connection) and fluxes (defining the normal bivectors to surfaces). The Poisson brackets between those variables, computed from the symplectic structure of general relativity, form a closed Lie algebra, called the holonomy-flux algebra. Its canonical quantization yields LQG and defines the unique diffeomorphism-covariant representation of those observables on quantum states (see [2-4] for a thorough presentation and [5-8] for a pedagogical overview).

> 圈量子引力聚焦于一组完备可观测量: 全纯和乐 (编码联络定义的空间点之间的有限平移) 和流 (定义曲面的法二矢量)。由广义相对论辛结构计算得到的这些变量之间的泊松括号构成闭合李代数，称为全纯和乐-流代数。对其进行正则量子化就得到圈量子引力，并给出这些可观测量在量子态上唯一的微分同胚协变表示 (全面介绍见文献 [2-4]，教学性综述见 [5-8])。

Let us underline an important subtlety of the formalism. The fluxes and holonomies are canonically conjugate variables. However, the fluxes satisfy second-class constraints, which reflect that they are entirely constructed from the tetrad fields even though they have more components. The standard LQG formulation solves those ”simplicity” constraints by choosing a specific gauge, called ”time gauge,” which sets the time-like normal vectors in internal space to the reference 4-vector $(1, 0, 0, 0)$ [9, 10] (see [11-13] for issues related to this gauge fixing). This condition breaks the local Lorentz symmetry down to a local SU(2) symmetry under spatial rotations, so standard LQG is formulated in terms of SU(2) holonomies and SU(2) fluxes, as we will use in the present chapter. It is nevertheless to keep in mind that a modern framework has been developed which explicitly solves the simplicity constraints without gauge fixing and restores the Lorentz gauge symmetry [14,15].

我们需要强调该形式体系一个重要的微妙之处。流和全纯和乐是正则共轭变量，但流满足第二类约束——这反映出尽管流拥有更多分量，它完全由标架场构造得到。标准圈量子引力表述通过选取特定规范 (称为“时间规范”) 解决了这些“简约性”约束，该规范将内空间的类时法矢固定为参考四矢量 $(1, 0, 0, 0)$ [9, 10] (该规范固定相关问题见 [11-13])。这个条件将局域洛伦兹对称性破缺为空间转动下的局域 SU(2) 对称性，因此标准圈量子引力采用 SU(2) 全纯和乐与 SU(2) 流表述，我们本章也沿用这一设定。但需要记住，目前已经发展出不做规范固定就显式求解简约性约束、恢复洛伦兹规范对称性的现代框架 [14,15]。

In this section, we define those quantum states of geometry in terms of spin networks. We recall their geometrical interpretation as discrete geometries and explain their re-interpretation as networks of entanglement between space-time points. In particular, we carefully distinguish the different types of degrees of freedom encoded in the spin network states and the entanglement they carry. This underlines the crucial role of two-dimensional boundaries in LQG and leads us to discuss the bulk-boundary relation and investigate holographic properties of spin networks.

本节中，我们用自旋网络定义这些几何量子态。我们会回顾它们作为离散几何的几何诠释，也会解释将它们重新诠释为时空点之间纠缠网络的观点。我们尤其会仔细区分自旋网络态编码的不同类型自由度，以及它们携带的纠缠。这一点凸显了二维边界在圈量子引力中的关键作用，并引导我们讨论体-边界关系，研究自旋网络的全息性质。

Wave-Functions and Spin Networks

波函数与自旋网络

Loop quantum gravity defines quantum states that capture the geometry of a finite number of space-time degrees of freedom. Indeed, as illustrated in Fig. 1, a state is defined by choosing a finite number of points on the 3D space manifold and a graph linking those points together and by considering a wave-function of the holonomies of the Ashtekar-Barbero connection along the links of the chosen graph. Those holonomies are group elements of the Lie group SU(2) and encode the transport, or change of 3D reference frame, from one point to another. The emphasis on using the transport between reference frames as the fundamental variable for the geometry hardcodes relativity and the notion of change of observers in the kinematics of theory.

圈量子引力定义了描述有限个时空自由度几何的量子态。正如图 1 所示，我们通过在三维空间流形上选取有限个点、连接这些点构成一张图，并考虑所选图各边上阿西特卡-巴贝罗联络的和乐，从而定义一个量子态。这些和乐是李群 SU(2) 的群元，编码了从一个点到另一个点的平移过程，即三维参考系的变化。将参考系之间的平移作为几何的基本变量，就在理论的运动学中固有地纳入了相对论和观测者变化的概念。

So let us consider a closed oriented graph Γ with E edges and V vertices. Focusing on a finite set of observables living on the graph - the holonomies along the graph links do not necessarily mean that there is no geometry away from the chosen graph but that the whole geometry of space is to be entirely reconstructed from the data living on the graph [16]. We consider gauge-invariant wave-functions depending on $\mathrm{SU}(2)$ group elements on every link or edge $e \in \Gamma$ while being invariant under the $\mathrm{SU}(2)$ action at every vertex $v \in \Gamma$:

我们现在来考虑一个封闭定向图 Γ，它包含 E 条边和 V 个顶点。聚焦于图上承载的有限可观测量集合并不意味着所选图之外不存在几何，而是说空间的整体几何都可以从图上承载的数据完整重构出来 [16]。我们考虑依赖于每条边 $e \in \Gamma$ 上 $\mathrm{SU}(2)$ 个群元的规范不变波函数，且该波函数在每个顶点 $v \in \Gamma$ 的 $\mathrm{SU}(2)$ 作用下保持不变:

$$\varphi\left(\{g_e\}_{e\in\Gamma}\right) = \varphi\left(\left\{h_{s(e)}g_e h_{t(e)}^{-1}\right\}_{e\in\Gamma}\right) \ \forall h_v \in \mathrm{SU}(2)^{\times V}, \tag{1}$$

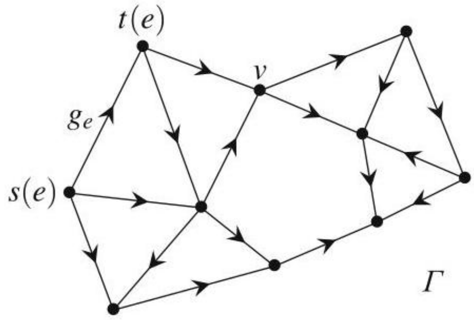

Fig. 1 Quantum states of geometry in loop quantum gravity are wave-functions of the SU(2) transport along the edges e of a graph Γ. The gauge invariance under local SU(2) transformations is imposed at every vertex v of the graph

图 1 圈量子引力中的几何量子态是图 Γ 各边 e 上 SU(2) 平移的波函数。在图的每个顶点 v 上都要求满足局域 SU(2) 变换下的规范不变性

where $s(e)$ stands for the source vertex of the oriented e and $t(e)$ the corresponding target vertex, as drawn in Fig. 1. The symmetry requirement implements the gauge invariance under local SU(2) transformations or equivalently local changes of 3D reference frame.

其中 $s(e)$ 是定向边 e 的源顶点，$t(e)$ 是对应边的目标顶点，如图 1 所示。这一对称性要求实现了局域 SU(2) 变换下的规范不变性，等价于三维参考系的局域变换不变性。

The Hilbert space of quantum states of geometry on that graph is then the L^2 space of gauge-invariant wave-functions provided with the Haar measure on the Lie group SU(2):

因此该图上几何量子态的希尔伯特空间就是规范不变波函数的 L^2 空间，配备 SU(2) 李群上的哈尔测度:

$$\mathcal{H}_\Gamma = L^2\left(\mathrm{SU}(2)^{\times E}/\mathrm{SU}(2)^{\times V}\right), \tag{2}$$

$$\langle \varphi \mid \widetilde{\varphi} \rangle_\Gamma = \int_{\mathrm{SU}(2)^{\times E}} \prod_e \mathrm{d}g_e \overline{\varphi\left(\{g_e\}_{e\in\Gamma}\right)} \widetilde{\varphi}\left(\{g_e\}_{e\in\Gamma}\right). \tag{3}$$

The full Hilbert space of loop quantum gravity is then defined as the sum of those individual spaces $\mathcal{H}_T$ over all graphs defined mathematically as a projective limit taking into account the inclusion of graphs into one another [17].

圈量子引力的全希尔伯特空间定义为所有图对应单个空间 $\mathcal{H}_T$ 的和，数学上定义为考虑图之间包含关系的投射极限 [17]。

An orthonormal basis of $\mathcal{H}_\Gamma$ is defined by using the Peter-Weyl theoreom and the Plancherel decomposition of functions on SU(2). This is the equivalent of a Fourier decomposition or more precisely the extension of the decomposition of functions on the two-sphere into spherical harmonics, where the Fourier modes are given by the Wigner matrices. This leads the spin network basis states $|\Gamma, j_e, I_v\rangle$, labeled by a spin $j_e \in \frac{\mathbb{N}}{2}$ on each edge e and an intertwiner I_v at every vertex v , as illustrated in Fig. 2:

利用彼得-外尔定理和 SU(2) 上函数的普朗歇尔分解可以得到 $\mathcal{H}_\Gamma$ 的一组标准正交基。这等价于傅里叶分解，更准确地说，它是二维球面上函数球谐分解的推广，其中傅里叶模由维格纳矩阵给出。由此我们得到自旋网络基态 $|\Gamma, j_e, I_v\rangle$ ，每个边 e 由一个自旋 $j_e \in \frac{\mathbb{N}}{2}$ 标记，每个顶点 v 由一个纠缠子 I_v 标记，如图 2 所示:

$$\mathcal{H}_\Gamma = \bigoplus_{\{j_e, I_v\}_{e,v\in\Gamma}} \mathbb{C}\,|\Gamma, j_e, I_v\rangle. \tag{4}$$

A spin j defines a irreducible representation of the Lie group SU(2). The associated Hilbert space, noted V^j , has dimension $d_j = (2j+1)$. Writing J^a with $a = x, y, z$ for the $\mathfrak{su}$ (2) Lie algebra generators, we use the usual basis of V^j diagonalizing both the $\mathfrak{su}$ (2) Casimir $\mathbf{J}^2 \equiv \delta_{ab} J^a J^b$ and the generator J^z and labeled by the spin j and the magnetic momentum m running by integer step from $-j$ to $+j$:

一个自旋 j 对应李群 SU(2) 的一个不可约表示。对应的希尔伯特空间记为 V^j ，维数为 $d_j = (2j+1)$ 。记 J^a ，其中 $a = x, y, z$ 是 $\mathfrak{su}$ (2) 李代数生成元，我们通常取 V^j 的一组基，该基同时对角化 $\mathfrak{su}$ (2) 的卡西米尔量 $\mathbf{J}^2 \equiv \delta_{ab} J^a J^b$ 和生成元 J^z ，由自旋 j 和磁矩 m 标记，磁矩以整数步长从 $-j$ 取到 $+j$:

$$V^j = \bigoplus_{-j\le m\le j} \mathbb{C}|j, m\rangle,\ d_j = \dim V^j = 2j+1, \tag{5}$$

$$\mathbf{J}^2\,|j, m\rangle = j\,(j+1)|\,j, m\rangle,\ J^z\,|j, m\rangle = m|\,j, m\rangle.$$

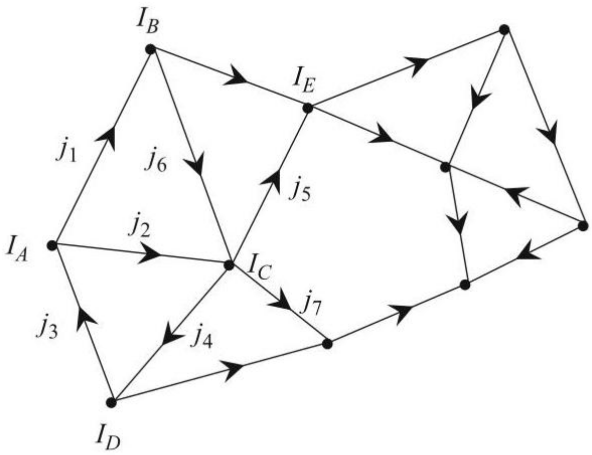

Fig. 2 A spin network, on a closed oriented graph Γ , is a basis state labeled with spins $j_e \in \frac{N}{2}$ on the graph edges and intertwiner states I_v on the graph vertices. Intertwiners are maps that commute with the $SU(2)$ action, between the tensor product of incoming spins and the tensor product of outgoing spins. They can be understood as singlet states. Here, I_A is a three-valent intertwiner from V^{j_3} to $V^{j_1} \otimes V^{j_2}$, while I_C is a five-valent intertwiner mapping $V^{j_2} \otimes V^{j_6}$ to $V^{j_4} \otimes V^{j_5} \otimes V^{j_7}$

图 2: 闭合定向图 Γ 上的自旋网络是基态，图的边标记自旋 $j_e \in \frac{N}{2}$ ，图的顶点标记缠结态 I_v 。缠结是与 $SU(2)$ 作用对易的映射，作用于入射自旋张量积与出射自旋张量积之间。它们可被理解为单态。此处 I_A 是从 V^{j_3} 到 $V^{j_1} \otimes V^{j_2}$ 的三价缠结，而 I_C 是将 $V^{j_2} \otimes V^{j_6}$ 映射为 $V^{j_4} \otimes V^{j_5} \otimes V^{j_7}$ 的五价缠结

An intertwiner I_v at the vertex v is a SU(2)-invariant map between the tensor product of the incoming spins and the tensor product of the outgoing spins, as illustrated in Fig. 2:

顶点 v 处的缠结 I_v 是入射自旋张量积与出射自旋张量积之间的 SU(2) 不变映射，如图 2 所示:

$$I_v : \bigotimes_{e|t(e)=v} V^{j_e} \to \bigotimes_{e|s(e)=v} V^{j_e}, \text{ such that } \forall h \in \mathrm{SU}(2), h \circ I_v = I_v \circ h. \tag{6}$$

Since the complex conjugate representation $\left(V^j\right)^*$ is isomorphic to V^j , intertwiners can be simply seen as singlet states, in the tensor product of all the spins living on the edges linked to v :

由于复共轭表示 $\left(V^j\right)^*$ 同构于 V^j ，缠结可简单看作关联到 v 的所有边所携带自旋的张量积中的单态:

(7)

$$I_v \in \mathscr{I}_v = \mathrm{Inv}_{\mathrm{SU}(2)}\Big[\bigotimes_{e|s(e)=v} V^{j_e} \otimes \bigotimes_{e|t(e)=v} (V^{j_e})^*\Big] \sim \mathrm{Inv}_{\mathrm{SU}(2)}\Big[\bigotimes_{e\ni v} V^{j_e}\Big].$$

By definition of an irreducible representation, bivalent intertwiners only exist if the two spins $j_1 = j_2$ are equal and are then unique. Trivalent intertwiners between three spins exist if and only if those spins satisfy triangular inequalities, $|j_1 - j_2| \leq j_3 \leq (j_1 + j_2)$, and are then unique: they are given by the Clebsh-Gordan coefficients. From valence 4 onwards, the intertwiner space grows in dimension, and they are multiple non-trivial intertwiner states. An interesting result from the representation theory of semi-simple Lie algebras is that higher-valent intertwiners can always be made from three-valent intertwiners by unfolding the vertex into a three-valent tree.

根据不可约表示的定义，双价缠结仅当两个自旋 $j_1 = j_2$ 相等时存在，且此时是唯一的。三个自旋之间的三价缠结存在当且仅当这些自旋满足三角不等式 $|j_1 - j_2| \leq j_3 \leq (j_1 + j_2)$ ，此时也唯一: 由克莱布希-高登系数给出。从四价开始，缠结空间的维度随价数增长，存在多个非平凡缠结态。半单李代数表示论的一个有趣结论是，高价缠结总能通过将顶点展开为三价树，由三价缠结构造得到。

Then the spin network wave-function is obtained by gluing the chosen inter-twiners at the vertices together connecting them by the Wigner matrices of the holonomies along the graph edges:

自旋网络波函数可通过如下方式得到: 将顶点处选定的缠结粘合在一起，再通过图边沿线移的维格纳矩阵将它们连接起来:

$$\varphi^{\{j_e, I_v\}}(\{g_e\}) = \sum_{m_e^{s,t}} \prod_e D^{j_e}_{m_e^t m_e^s}(g_e) \prod_v \left\langle \bigotimes_{e|s(e)=v} j_e m_e^s \,|I_v|\, \bigotimes_{e|t(e)=v} j_e m_e^t \right\rangle, \tag{8}$$

where $D^j_{mm'}(g) = \langle j, m|g|\, j, m'\rangle$ are the matrix elements of the Wigner matrix $D^j(g)$ representing the SU (2) group element g in the spin- j representation in the magnetic moment basis.

其中 $D^j_{mm'}(g) = \langle j, m|g|\, j, m'\rangle$ 是维格纳矩阵 $D^j(g)$ 的矩阵元，该矩阵表示磁矩基下自旋- j 表示中的 SU (2) 群元素 g 。

In the geometrical interpretation of loop quantum gravity [18-20], intertwiners represent excitations of the 3D volume, while spins give the quanta of area of the interface gluing neighboring chunks of volume (in other words, the cross section). This is further validated by the geometrical interpretation of intertwiners as quantized convex polyhedra [21-23] with the spins giving the area of the polyhedra' faces. From this perspective, spin networks are interpreted as discrete geometries or more precisely the quantization of 3D twisted geometries [24]. Twisted geometries are an extension of 3D Regge geometries to discontinuous 4D embedding [25].

在圈量子引力的几何诠释中 [18-20]，缠结表示三维体积的激发，而自旋给出粘合相邻体积块 (即截面) 的界面的面积量子。这一点进一步被缠结的几何诠释佐证: 缠结对应量子化凸多面体 [21-23]，自旋给出多面体每个面的面积。从这个角度看，自旋网络被诠释为离散几何，更准确地说，是三维扭曲几何的量子化 [24]。扭曲几何是三维里奇几何对不连续四维嵌入的扩展 [25]。

The goal of the present chapter is to introduce the tools to interpret the geometry of spin network in light of quantum information. In particular, we explain how the basic components of the quantum geometry - the spins and intertwiners - are reflected in the entanglement.

本章的目标是引入相关工具，从量子信息的角度诠释自旋网络的几何。我们尤其会阐释量子几何的基本组分——自旋与缠结——如何体现在纠缠中。

Geometry as a Network of Entanglement

几何作为纠缠网络

Now, LQG's quantum states of geometry are spin networks, built from gluing quanta of 3D volume, the intertwiners, together through quanta of 2D area, the spin-network edges. This gluing is not an innocent operation; it actually creates entanglement between neighboring quanta of 3D volume. So spin networks can be thought of as networks of entanglement.

目前，LQG 的几何量子态是自旋网络，由三维体积量子 (即 intertwiners，交缠子) 通过二维面积量子 (即自旋网络边) 粘合构建而成。这种粘合并非无足轻重的操作；它实际上会在相邻的三维体积量子之间产生纠缠。因此自旋网络可以被视为纠缠网络。

In order to make this picture concrete, let us look more closely at the entanglement across a spin network edge, as depicted in Fig. 3. There are three different notions of entanglement, depending of the objects on which we focus:

为了让这一图像更具体，我们来更仔细地分析自旋网络边的纠缠，如图 3 所示。根据我们关注的对象不同，共有三种不同的纠缠概念:

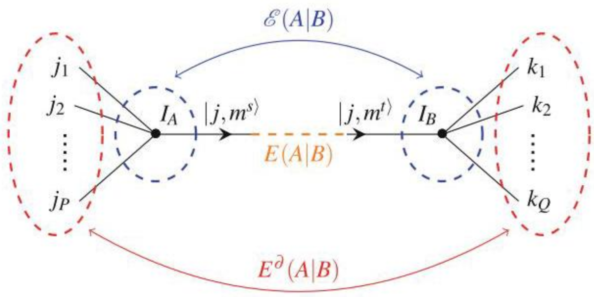

Fig. 3 We distinguish the link entanglement $E(A \mid B)$ between the spin states living at the two ends of the spin network edge, the intertwiner entanglement $\mathcal{E}(A \mid B)$ between the two intertwiners living at the vertices A and B , and the boundary spin entanglement $E^\partial(A \mid B)$ between the spins living on the boundary of the spin network region formed by merging the two 3D regions A and B

图 3 我们区分三种纠缠: 位于自旋网络边两端的自旋态之间的链接纠缠 $E(A \mid B)$ 、位于顶点 A 和 B 处的两个交缠子之间的交缠子纠缠 $\mathcal{E}(A \mid B)$ ，以及合并 A 和 B 两个三维区域后得到的自旋网络区域边界上的自旋态之间的边界自旋纠缠 $E^\partial(A \mid B)$

- The link entanglement between the two spins that are directly glued together by the link

- 由链接直接粘合的两个自旋之间的链接纠缠

- The intertwiner entanglement between the two intertwiners living at the vertices glued by the link

- 位于被该链接粘合的顶点处的两个交缠子之间的交缠子纠缠

- The boundary spin entanglement between the other spins around those vertices

- 顶点处其余自旋之间的边界自旋纠缠

Let us underline that spin states $|j, m\rangle$ live on the half-edges of the spin networks. They are not SU(2)-gauge invariant object, and they are in fact summed over in the definition of the spin network wave-function. From that point of view, the link entanglement does not measure the quantum entanglement between physical degrees of freedom. On the other hand, intertwiners are legitimate gauge-invariant objects and reveal quantum correlations between the two spin network vertices. However, they do not reflect at all whether an actual link exists between those two vertices or not. Nevertheless, putting these two notions together leads to the boundary spin entanglement. Indeed, as we will prove below, the boundary spin entanglement is indeed the sum of the link entanglement and intertwiner entanglement.

需要强调的是，自旋态 $|j, m\rangle$ 位于自旋网络的半边上。它们不是 SU(2) 规范不变的对象，实际上在定义自旋网络波函数时会被求和积分。从这个角度看，链接纠缠衡量的不是物理自由度之间的量子纠缠。另一方面，交缠子是合规的规范不变对象，能够反映两个自旋网络顶点之间的量子关联。但交缠子纠缠完全无法反映两个顶点之间是否实际存在链接。将这两个概念结合就得到了边界自旋纠缠，我们会在下文中证明，边界自旋纠缠确实等于链接纠缠与交缠子纠缠之和。

While glued spin states along spin network edges are not physical degrees of freedom, spin states living on cut links have a different status. Indeed, they are interpreted as living on the boundary of the spin network. Here, considering two vertices glued by one edge, the spins living on all the other links attached to the vertices constitute the boundary of that region of space. They become legitimate degrees of freedom (see, e.g., [26] for a reformulation of boundary modes as would-be-gauge degrees of freedom in gauge theories, relevant for LQG), and the boundary spin entanglement is a tangible measure of the correlations between them.

沿自旋网络边粘合的自旋态不属于物理自由度，但截断链接上的自旋态地位不同: 它们被解释为位于自旋网络的边界上。此处，对于被一条边粘合的两个顶点，附着在这两个顶点上的所有其他链接上的自旋构成了该空间区域的边界，这些自旋是合规的物理自由度 (例如，参见文献 [26]，该文献将边界模式重新表述为规范理论中的拟规范自由度，对 LQG 适用)，边界自旋纠缠就是它们之间关联的可观测度量。

Link Entanglement

链接纠缠

As drawn in Fig. 4, a spin network link e carries two spin states, one on each end of the link. These two spin states are related by the holonomy $g \in \mathrm{SU}(2)$ running along the link. Let us assume that the link is dressed with a fixed spin j . Then the state associated to the link lives in $V^j \otimes \left(V^j\right)^* = \mathrm{End}\left[V^j\right]$. In the case that the holonomy is known, then the state is

如图 4 所示,自旋网络的链接 e 携带两个自旋态,链接两端各有一个。这两个自旋态由沿链接传播的和乐 $g \in \mathrm{SU}(2)$ 关联。我们假设该链接带有固定自旋 j , 那么链接对应的态位于 $V^j \otimes \left(V^j\right)^* = \mathrm{End}\left[V^j\right]$ 中。若和乐已知，该态可写为

$$\Phi^{(j,g)} = \frac{1}{\sqrt{d_j}} \sum_{m^s, m^t} D^j_{m^t m^s}(g) \, |j, m^t\rangle\langle j, m^s| \in V^j \otimes (V^j)^*. \tag{9}$$

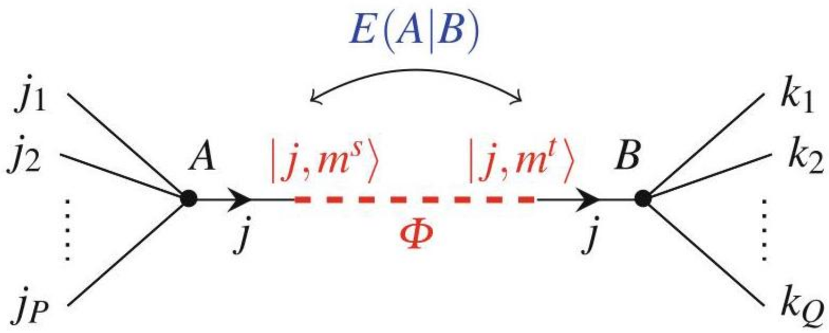

Fig. 4 The two spin network vertices A and B are glued along a link carrying the spin j . The link entanglement $E(A \mid B)$ along that spin network edge is the entanglement between the source spin $|j, m^s\rangle$ and the target spin $|j, m^t\rangle$ defined by the chosen link state Φ , which controls the gluing. This link state links in $V^j \otimes (V^j)^*$ and represents the superposition of SU (2) holonomies transporting the local reference frame from A to B

图 4 两个自旋网络顶点 A 和 B 沿一条携带自旋 j 的链接粘合。该自旋网络边上的链接纠缠 $E(A \mid B)$ 是源自旋 $|j, m^s\rangle$ 与靶自旋 $|j, m^t\rangle$ 之间由选定链接态 Φ 定义的纠缠，这个链接态控制着粘合过程。该链接态关联 $V^j \otimes (V^j)^*$ ，代表 SU (2) 个和乐的叠加，这些和乐将局域参考系从 A 传输到 B

Since the Wigner matrices are unitary, this state (Using the ismorphism between V^j and $(V^j)^*$, we can map this into a state in $V^j \otimes V^j$:

由于维格纳矩阵是幺正的，利用 V^j 和 $(V^j)^*$ 之间的同构，我们可以将该态映射为 $V^j \otimes V^j$ 中的态:

$$\Phi^g_e \mapsto \frac{1}{\sqrt{d_j}} \sum_{m^s, m^t} (-1)^{j-m^s} D^j_{m^t m^s}(g) \, |j, m^t\rangle \otimes |j, -m^s\rangle \in V^j \otimes V^j. \tag{10}$$

In the special case where the holonomy is trivial along the link, $g = \mathbb{I}$, then the Wigner matrix is the identity, and the link state reduces to the standard singlet state $\sum_m (-1)^{j-m} |j, m\rangle \otimes | j, -m\rangle \in \mathrm{Inv}_{\mathrm{SU}(2)} \left[V^j \otimes V^j\right]$.) is normalized. This clearly is a maximally entangled state between the source and target. To check this, we compute the reduced density matrix, for instance, for the target spin, which simplifies due the unitarity of the Wigner matrix:

当沿链接的和乐是平凡和乐，即 $g = \mathbb{I}$ 时，维格纳矩阵是单位矩阵，链接态退化为标准单重态 $\sum_m (-1)^{j-m} |j, m\rangle \otimes | j, -m\rangle \in \mathrm{Inv}_{\mathrm{SU}(2)} \left[V^j \otimes V^j\right]$ 。) 该态是归一化的，显然是源自旋与靶自旋之间的最大纠缠态。为验证这一点，我们计算约化密度矩阵，例如靶自旋的约化密度矩阵，利用维格纳矩阵的幺正性它可以化简为:

$$\rho_t^{(j,g)} = \frac{1}{d_j} \sum_{m^t} |j, m^t\rangle\langle j, m^t| = \mathbb{I}_j, \tag{11}$$

where $\mathbb{I}_j$ is the identity on the Hilbert space V^j . This reduced density matrix does not depend on the holonomy g and is always the totally mixed state, with maximal entropy:

其中 $\mathbb{I}_j$ 是希尔伯特空间 V^j 上的单位算符。该约化密度矩阵不依赖于和乐 g，始终是全混合态，具有最大熵:

$$E^{(j,g)}(s \mid t) = -\operatorname{Tr} \rho_t^{(j,g)} \ln \rho_t^{(j,g)} = \ln d_j = \ln(2j+1)\,. \tag{12}$$

The link entanglement is non-zero as soon as the spin carried by the link does not vanish $j \neq 0$. In fact, considering a vanishing spin $j = 0$ is completely equivalent to having no link. Thus having a non-trivial link between two vertices in a spin network means having a non-trivial entanglement. This leads to the picture of a spin network as a graph whose links represent a (maximal) entanglement between spin states or, in simpler terms, an entanglement graph [27, 28].

只要链接携带的自旋不为零 $j \neq 0$，链接纠缠就非零。事实上，自旋为零 $j = 0$ 等价于不存在该链接。因此，自旋网络中两个顶点之间存在非平凡链接，就意味着存在非平凡纠缠。由此我们可以将自旋网络理解为一张图: 图的链接代表自旋态之间的 (最大) 纠缠，简单来说就是一张纠缠图 [27, 28]。

Nevertheless, in general, the holonomy and the spins are not fixed. They are distributed according to the considered graph wave-function. This is appropriately described by a link state that depends on a probability amplitude for the holonomy $\phi(g) \in L^2(\mathrm{SU}(2))$. By decomposing this probability amplitude into spins by the Peter-Weyl theorem, one translates it into the corresponding link state:

但一般情况下，和乐和自旋并不是固定的，它们按照给定图波函数分布，这可以通过依赖和乐概率幅 $\phi(g) \in L^2(\mathrm{SU}(2))$ 的链接态来恰当地描述。利用彼得-外尔定理将该概率幅按自旋分解后，我们就能将其转换为对应的链接态:

$$\phi(g) = \sum_{j,m^s,m^t} \sqrt{d_j}\phi^j_{m^t m^s} D^j_{m^t m^s}(g) \mapsto \Phi = \sum_{j,m^s,m^t} \phi^j_{m^t m^s} \left|j, m^t\right\rangle \langle j, m^s|$$

(13)

$$\text{with the normalization} \int \mathrm{d}g|\phi(g)|^2 = 1 = \sum_j \operatorname{Tr}_j \phi^j (\phi^j)^\dagger = \langle \Phi \mid \Phi \rangle\,. \tag{14}$$

The reduced density matrix for the target spin reads:

靶自旋的约化密度矩阵为:

$$\rho_t^\phi = \operatorname{Tr}_s |\Phi\rangle\langle\Phi| = \sum_{j,m^t\widetilde{m}^t} \left[\phi^j (\phi^j)^\dagger\right]_{m^t\widetilde{m}^t} \left|j, m^t\right\rangle \langle j, \widetilde{m}^t| \in \operatorname{End}\left[\bigoplus_j V^j\right].\Big|$$

(15)

This matrix is diagonal by block, each block corresponding to a spin j , and its Von Neumann entropy gives the entanglement carried by the link:

该矩阵是分块对角的，每一块对应一个自旋 j，其冯·诺依曼熵就是该链接携带的纠缠:

$$E_\Phi(s \mid t) = -\sum_j \mathrm{Tr}\left(\rho_j \ln \rho_j\right) \quad \text{with } \rho_j = \phi^j \left(\phi^j\right)^\dagger. \tag{16}$$

One can fix the spin by requiring that the probability amplitude ϕ for the holonomy carries a single spin j_0 , i.e., that the matrices ϕ_j vanish except for $j = j_0$. In this case, the entanglement sum formula reduces to a single term:

我们可以通过要求全纯性的概率幅 ϕ 仅携带单个自旋 j_0 来固定自旋，即矩阵 ϕ_j 除 $j = j_0$ 外均为零。这种情况下，纠缠和公式简化为单独一项:

$$\phi_j \propto \delta_{j j_0} \Rightarrow E_\Phi(s \mid t) = -\mathrm{Tr}\left(\rho_{j_0} \ln \rho_{j_0}\right) \quad \text{with } \mathrm{Tr}\, \rho_{j_0} = 1. \tag{17}$$

Then the entanglement is bounded by the value of the spin $E_\Phi(s \mid t) \leq \ln(2j_0 + 1)$. The maximal value is reached when ρ_{j_0} is the totally mixed state, in which case we recover the result obtained earlier for fixed spin and holonomy:

此时纠缠由自旋 $E_\Phi(s \mid t) \leq \ln(2j_0 + 1)$ 的值界定。当 ρ_{j_0} 为全混合态时取到最大值，该情况下我们可以得到此前固定自旋与全纯性时得到的结果:

$$\rho_j = \delta_{j j_0} \frac{1}{d_{j_0}} \mathbb{I}_{j_0} \Rightarrow E_\Phi(s \mid t) = \ln(2j_0 + 1). \tag{18}$$

This means that, for a fixed spin, non-maximal entanglement along a spin network link comes from quantum superpositions of holonomies.

这意味着，对于固定自旋，自旋网络连线上的非最大纠缠来源于全纯性的量子叠加。

Intertwiner Entanglement vs. Boundary Spin Entanglement

缠结子纠缠 vs 边界自旋纠缠

The link entanglement along a spin network edge only reflects the act of gluing two vertices along that edge. It does not provide a measure of the quantum correlations between the quanta of 3D geometry living at the two vertices linked by that edge. Let us thus focus on two neighboring vertices of a spin network, A and B , as in Fig. 5, linked by a single edge oriented from A to B and decorated with a fixed given spin j . The other legs attached to the vertex A are decorated with spins $j_1, \ldots, j_P$, while the other edges attached to B carry the spins $k_1, \ldots, k_Q$. Having fixed all the spins, the only remaining freedom is the choice of intertwiner states. Indeed the Hilbert space of spin network states for this configuration at fixed spins is

自旋网络边上的链路纠缠仅反映沿该边粘合两个顶点的操作。它无法度量由该边连接的两个顶点上的三维几何量子之间的量子关联。因此我们聚焦于自旋网络的两个相邻顶点 A 和 B ，如图 5 所示，二者由一条方向从 A 指向 B 的单一边连接，该边带有确定自旋 j 。连接到顶点 A 的其他支腿带有自旋 $j_1, \ldots, j_P$ ，而连接到 B 的其他边带有自旋 $k_1, \ldots, k_Q$ 。固定所有自旋后，唯一剩余的自由度就是选择缠结子态。实际上，该构型在固定自旋下的自旋网络态希尔伯特空间为

$$\mathcal{H}_{AB} = \mathcal{I}_A \otimes \mathcal{I}_B \tag{19}$$

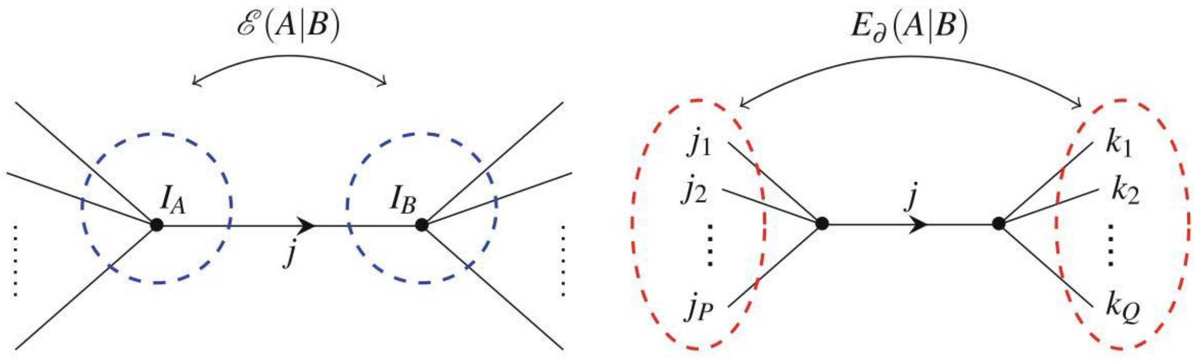

Fig. 5 Intertwiner entanglement entropy (on the left) versus boundary spin state entanglement (on the right): we distinguish the entanglement $\mathcal{E}(A \mid B)$ between the intertwiner states at the two nodes in blue and the entanglement $E_\partial(A \mid B)$ they induce on the boundary spins in red

图 5 缠结子纠缠熵 (左) 与边界自旋态纠缠 (右): 我们用蓝色区分两个节点上缠结子态之间的纠缠 $\mathcal{E}(A \mid B)$， 用红色区分它们在边界自旋上诱导的纠缠 $E_\partial(A \mid B)$

where $\mathcal{I}_A$ and $\mathcal{I}_B$ are the spaces of intertwiners attached to the two nodes:

其中 $\mathcal{I}_A$ 和 $\mathcal{I}_B$ 是附属于两个节点的缠结子空间:

$$\mathcal{I}_A = \mathrm{Inv}_{\mathrm{SU}(2)}\left[V^{j_1} \otimes \ldots \otimes V^{j_P} \otimes \left(V^j\right)^*\right], \ \mathcal{I}_B = \mathrm{Inv}_{\mathrm{SU}(2)}\left[V^{k_1} \otimes \ldots \otimes V^{k_Q} \otimes V^j\right].$$

(20)

We assumed that all the boundary spins j_p and k_q are incoming onto A and B .

我们假设所有边界自旋 j_p 和 k_q 都是入射到 A 和 B 的。

Considering a pure state in $\mathcal{H}_{AB}$, we define the intertwiner entanglement $\mathcal{E}(A \mid B)$ between A and B as the Von Neumann entropy of the reduced density matrices, obtained by tracing over $\mathcal{I}_A$ or $\mathcal{I}_B$. Let’s be more explicit by choosing an orthonormal basis for the intertwiners at the two nodes:

考虑 $\mathcal{H}_{AB}$ 中的一个纯态，我们将 A 与 B 之间的缠结子纠缠 $\mathcal{E}(A \mid B)$ 定义为约化密度矩阵的冯·诺依曼熵，该约化密度矩阵通过对 $\mathcal{I}_A$ 或 $\mathcal{I}_B$ 求迹得到。我们为两个节点的缠结子选择一组标准正交基，来给出更明确的表述:

$$I_A^\alpha \in \mathcal{I}_A = \mathrm{Inv}\left[\bigotimes_{p=1}^{P} V^{j_p} \otimes \left(V^j\right)^*\right], \ \left\langle I_A^\alpha \mid I_A^{\tilde{\alpha}}\right\rangle = \delta_{\alpha\tilde{\alpha}}, \tag{21}$$

$$I_B^\beta \in \mathcal{I}_B = \mathrm{Inv}\left[\bigotimes_{q=1}^{Q} V^{k_q} \otimes V^j\right], \ \left\langle I_B^\beta \mid I_B^{\tilde{\beta}}\right\rangle = \delta_{\beta\tilde{\beta}}, \tag{22}$$

For an arbitrary normalized spin network state $I \in \mathcal{H}_{AB} = \mathcal{I}_A \otimes \mathcal{I}_B$,

对任意归一化自旋网络态 $I \in \mathcal{H}_{AB} = \mathcal{I}_A \otimes \mathcal{I}_B$ ，

$$\left|I\right\rangle = \sum_{\alpha\beta} \psi_{\alpha\beta} \left|I_A^\alpha\right\rangle \otimes \left|I_B^\beta\right\rangle, \; \langle I \mid I\rangle = \mathrm{Tr}\, \psi\psi^\dagger = 1, \tag{23}$$

we compute the reduced density matrix:

我们计算约化密度矩阵:

$$\rho_A = \mathrm{Tr}_B\, |I\rangle\langle I| = \sum_{\alpha,\tilde{\alpha}} \left(\psi\psi^\dagger\right)_{\alpha\tilde{\alpha}} \left|I_A^\alpha\right\rangle \left\langle I_A^{\tilde{\alpha}}\right|, \; \mathrm{Tr}\, \rho_A = 1, \tag{24}$$

from which we get the intertwiner entanglement:

由此我们得到缠结子纠缠:

$$\mathcal{E}_I\,(A \mid B) = -\,\mathrm{Tr}\, \psi\psi^\dagger \ln \psi\psi^\dagger. \tag{25}$$

Note that, even though the coefficient matrix ψ is a priori not a square matrix, the matrix $\psi\psi^\dagger$ is always a square matrix. This entanglement entropy is bounded by the dimension of the intertwiner spaces:

注意，即使系数矩阵 ψ 先验不是方阵，矩阵 $\psi\psi^\dagger$ 也始终是方阵。该纠缠熵有如下由缠结子空间维度给出的界:

$$\mathcal{E}_I\,(A \mid B) \leq \min\left(\ln \dim \mathcal{I}_A, \ln \dim \mathcal{I}_B\right). \tag{26}$$

Another way to look at the regions A and B , once glued by the intermediate edge, is to group them into a single region AB . The edge between A and B , carrying the spin j , is now within this coarser region AB , while the remaining edges carrying the spins $j_1, \dots, j_P, k_1, \dots, k_Q$ are all on its boundary. Then the Hilbert space of boundary spin states is

另一种研究区域 A 和 B 的视角是，在二者通过中间边粘合后，将它们组合为单个区域 AB 。连接 A 和 B 、带有自旋 j 的边现在处于这个更粗粒度的区域 AB 内部，而带有自旋 $j_1, \dots, j_P, k_1, \dots, k_Q$ 的剩余边全部位于该区域的边界上。此时边界自旋态的希尔伯特空间为

$$H_{AB}^{\partial} = \underbrace{\left(V^{j_1} \otimes \dots \otimes V^{j_P}\right)}_{H_A^{\setminus(AB)}} \otimes \underbrace{\left(V^{k_1} \otimes \dots \otimes V^{k_Q}\right)}_{H_B^{\setminus(AB)}}, \tag{27}$$

which is clearly not the same Hilbert space as $\mathcal{H}_{AB}$. We write $H_A^{\setminus(AB)} = \bigotimes_p V^{j_p}$ for the Hilbert space tensor product of all the spins attached to the vertex A but the one living on the edge (AB) linking the two vertices, similarly for B . The natural bipartite splitting of this boundary Hilbert space leads to a notion of boundary spin entanglement, a priori different from the intertwiner entanglement. As shown in [29], these two notions are nevertheless related. In order to make this explicit, we write the boundary spin states obtained from a spin network configuration by gluing the two vertices A and B by a link state of determined spin j :

它显然与 $\mathcal{H}_{AB}$ 不是同一个希尔伯特空间。我们记 $H_A^{\setminus(AB)}=\bigotimes_p V^{j_p}$ 为附属于顶点 A 、除去连接两个顶点的边 (AB) 上的自旋之外，所有自旋的希尔伯特空间张量积，B 同理。该边界希尔伯特空间自然的二分分裂引出了边界自旋纠缠的概念，它先验上不同于缠结子纠缠。但正如文献 [29] 所示，这两个概念存在关联。为了明确这一点，我们写出由以下自旋网络构型得到的边界自旋态: 将两个顶点 A 和 B 通过一个带有确定自旋 j 的链路态粘合:

Definition 1 (Gluing map from intertwiners to boundary spins). For a factorized spin network state $I_A\otimes I_B\in\mathcal{H}_{AB}$ with decoupled intertwiner states at the two vertices A and B , and a normalized link state Φ of fixed spin j ,

定义 1(从纠缠基到边界自旋的粘合映射)。对于在两个顶点 A 和 B 处具有退耦纠缠基态的因子化自旋网络态 $I_A\otimes I_B\in\mathcal{H}_{AB}$ ，以及固定自旋 j 的归一化链路态 Φ ,

$$\Phi=\sum_{a,b}\phi^j_{ba}\,|j,b\rangle\langle j,a|\ \in V^j\otimes\left(V^j\right)^*,\ \langle\Phi\mid\Phi\rangle=\operatorname{Tr}\phi^j\left(\phi^j\right)^\dagger=1, \tag{28}$$

we define the corresponding boundary spin state in H^∂_{AB} obtained by gluing the two intertwiners through this link state as

我们定义在 H^∂_{AB} 中通过该链路态粘合两个纠缠基得到的对应边界自旋态为

$$\mathcal{P}_\Phi\left[I_A\otimes I_B\right]=\langle\Phi\mid I_A\otimes I_B\rangle_{j\otimes j^*}=d_j\sum_{a,b}\phi^j_{ba}I_A^a\otimes I_B^b\in H^\partial_{AB}, \tag{29}$$

where $d_j=(2j+1)$ is a normalization factor. The notations I_A^a and I_B^b correspond to the projection of the intertwiner states on fixed magnetic moment of the spin j component:

其中 $d_j=(2j+1)$ 是归一化因子。记号 I_A^a 和 I_B^b 对应纠缠基态在自旋 j 分量的固定磁矩上的投影:

$$I_A=\sum_a I_A^a\otimes\langle j,a\,|\in\mathcal{I}_A\ \text{ with } I_A^a\in H_A^{\setminus(AB)} \tag{30}$$

$$I_B=\sum_b I_B^b\otimes|j,b\rangle\in\mathcal{I}_B\ \text{ with } I_B^b\in H_B^{\setminus(AB)} \tag{31}$$

The map $\mathcal{P}_\Phi$ is then extended to the whole Hilbert space $\mathcal{I}_A\otimes\mathcal{I}_B$ by linearity.

随后通过线性性将映射 $\mathcal{P}_\Phi$ 延拓到整个希尔伯特空间 $\mathcal{I}_A\otimes\mathcal{I}_B$ 。

A special case is when the holonomy along the linking edge is fixed $g\in\mathrm{SU}(2)$. The resulting gluing map $\mathcal{P}_g$ then reads:

一种特殊情况是连接边的和乐固定 $g\in\mathrm{SU}(2)$ ，此时得到的粘合映射 $\mathcal{P}_g$ 形如:

$$\Phi_g=\frac{1}{\sqrt{d_j}}\sum_{a,b}D^j_{ba}(g)\,|j,b\rangle\langle j,a|, \tag{32}$$

$$\mathcal{P}_g : I_A \otimes I_B \in \mathcal{H}_{AB} \mapsto \sqrt{d_j} \sum_{a,b} D^j_{ba}(g)\, I^a_A \otimes I^b_B \in H^\partial_{AB}. \tag{33}$$

The trivial gluing corresponds to $g = \mathbb{I}$, for which $\mathcal{P}_{\mathbb{I}}[I_A \otimes I_B] = \sqrt{d_j} \sum_a I^a_A \otimes I^a_B$.

平凡粘合对应 $g = \mathbb{I}$ ，此时满足 $\mathcal{P}_{\mathbb{I}}[I_A \otimes I_B] = \sqrt{d_j} \sum_a I^a_A \otimes I^a_B$ 。

Proposition 1 (Entanglement of Intertwiner Superpositions:). Let $I \in \mathcal{H}_{AB} = \mathcal{I}_A \otimes \mathcal{I}_B$ be an arbitrary normalized superposition of tensor product of intertwiner states. Then the corresponding boundary state $\mathcal{P}_\Phi[I] \in H^\partial_{AB}$, obtained by gluing the intertwiners along a link state of fixed spin- j , carries an entanglement simply equal to the sum of the intertwiner entanglement of I and of the link entanglement of Φ :

命题 1(纠缠基叠加的纠缠性质:)。设 $I \in \mathcal{H}_{AB} = \mathcal{I}_A \otimes \mathcal{I}_B$ 是纠缠基态张量积的任意归一化叠加，那么通过固定自旋- j 的链路态粘合纠缠基得到的对应边界态 $\mathcal{P}_\Phi[I] \in H^\partial_{AB}$ ，其纠缠恰好等于 I 的纠缠基纠缠与 Φ 的链路纠缠之和:

$$E^\partial_{\mathcal{P}_\Phi[I]}(A \mid B) = \mathcal{E}_I(A \mid B) + E_\Phi(A \mid B). \tag{34}$$

Proof. For an arbitrary normalized spin network state $I \in \mathcal{H}_{AB} = \mathcal{I}_A \otimes \mathcal{I}_B$,

证明。对于任意归一化自旋网络态 $I \in \mathcal{H}_{AB} = \mathcal{I}_A \otimes \mathcal{I}_B$,

$$\Big| I\rangle = \sum_{\alpha\beta} \psi_{\alpha\beta} \Big| I^\alpha_A\rangle \otimes \big| I^\beta_B\big\rangle, \ \langle I \mid I\rangle = \operatorname{Tr} \psi\psi^\dagger = 1, \tag{35}$$

and a normalized link state of fixed spin j ,

以及固定自旋 j 的归一化链路态,

$$\Phi = \sum_{a,b} \phi^j_{ba} \,|j,b\rangle\langle j,a| \ \in V^j \otimes \left(V^j\right)^*, \ \langle \Phi \mid \Phi\rangle = \operatorname{Tr} \phi^j \left(\phi^j\right)^\dagger = 1, \tag{36}$$

the boundary spin state resulting from gluing the intertwiners at A and B using the link state Φ_S

通过链路态 Φ_S 粘合 A 和 B 处的纠缠基得到的边界自旋态

$$\mathcal{P}_\Phi[I] = d_j \sum_{\alpha\beta} \psi_{\alpha\beta} \mathcal{P}_\Phi \left[I^\alpha_A \otimes I^\beta_B \right] = d_j \sum_{\alpha\beta} \sum_{a,b} \psi_{\alpha\beta} \phi^j_{ba} I^{\alpha,a}_A \otimes I^{\beta,b}_B, \tag{37}$$

with the notation $I^{\alpha,a}_A = \left(I^\alpha_A\right)^a \in H^{\setminus(AB)}_A = \bigotimes_{p=1}^P V^{j_p}$. The key property of these intertwiner components is that they are orthonormal:

满足记号 $I^{\alpha,a}_A = \left(I^\alpha_A\right)^a \in H^{\setminus(AB)}_A = \bigotimes_{p=1}^P V^{j_p}$ 。这些纠缠基分量的核心性质是它们标准正交:

$$\left\langle I_A^{\alpha,a} \mid I_A^{\tilde{\alpha},\tilde{a}} \right\rangle = \frac{1}{d_j} \delta_{\alpha,\tilde{\alpha}} \delta_{a,\tilde{a}}. \tag{38}$$

Indeed, each intertwiner basis label a defines an embedding of the tensor product $\bigotimes_{p=1}^{P} V^{j_p}$ into the single spin space V^j , i.e., a channel recoupling the spins $j_1, \dots, j_P$ into the spin j . This implies that projecting the intertwiner on different magnetic moments a leads to orthogonal states in $V^j \hookrightarrow \bigotimes_{i=p}^{P} V^{j_p}$. The same holds for the node B . This allows to check explicitly that the boundary spin state is normalized, $\langle \mathcal{P}_\Phi [I] \mid \mathcal{P}_\Phi [I] \rangle = 1$. We can then compute the reduced density on the boundary spins of A , as an endomorphism of $H_A^{\setminus (AB)} = \bigotimes_{i=p}^{P} V^{j_p}$:

事实上，每个缠结基标签 a 都定义了张量积 $\bigotimes_{p=1}^{P} V^{j_p}$ 到单自旋空间 V^j 的嵌入，也就是一个将自旋 $j_1, \dots, j_P$ 重新耦合到自旋 j 的通道。这意味着将缠结投影到不同磁矩 a 会得到 $V^j \hookrightarrow \bigotimes_{i=p}^{P} V^{j_p}$ 中的正交态。对节点 B 而言同理。这可以显式验证边界自旋态是归一化的，即 $\langle \mathcal{P}_\Phi [I] \mid \mathcal{P}_\Phi [I] \rangle = 1$ 。随后我们可以计算 A 边界自旋上的约化密度矩阵，作为 $H_A^{\setminus (AB)} = \bigotimes_{i=p}^{P} V^{j_p}$ 的自同态:

$$\rho_A^\partial = \sum_{\alpha\tilde{\alpha}} \sum_{a,\tilde{a}} \left(\psi\psi^\dagger\right)_{\alpha\tilde{\alpha}} \left(\phi^{j\dagger}\phi^j\right)_{\tilde{a}a} d_j \left| I_A^{\alpha,a} \right\rangle \left\langle I_A^{\tilde{\alpha},\tilde{a}} \right|, \ \mathrm{Tr}\, \rho_A^\partial = 1. \tag{39}$$

This directly gives the spin state entanglement:

由此可直接得到自旋态纠缠:

$$E^\partial_{\mathcal{P}_\phi[I]} (A \mid B) = -\left(\mathrm{Tr}\, \phi^\dagger \phi \ln \phi^\dagger \phi\right) - \left(\mathrm{Tr}\, \psi\psi^\dagger \ln \psi\psi^\dagger\right) = \mathcal{E}_I (A \mid B) + E_\Phi (A \mid B)$$

(40)

thus concluding the proof of this proposition extending the results of [29].

至此完成了推广文献 [29] 结果的该命题的证明。

A direct consequence of this proposition is its application to pure tensor product states, which do not carry any intertwiner entanglement, $E (A \mid B) = 0$, in which case the boundary spin entanglement $E^\partial (A \mid B)$ exactly reduces to the link entanglement. In particular:

该命题的一个直接推论是它对不携带任何缠结纠缠的纯张量积态的应用，即 $E (A \mid B) = 0$ ，此时边界自旋纠缠 $E^\partial (A \mid B)$ 恰好退化为链路纠缠。特别地:

Corollary 1 (Factorized Intertwiner States). Let $I_A \otimes I_B \in \mathcal{H}_{AB} = \mathcal{I}_A \otimes \mathcal{I}_B$ be a normalized intertwiner tensor product state. The resulting boundary spin entanglement resulting from gluing A and B with and edge with fixed j and given holonomy $g \in \mathrm{SU}(2)$ only depends on the spin j and does not depend on the holonomy g :

推论 1(分解缠结态)。设 $I_A \otimes I_B \in \mathcal{H}_{AB} = \mathcal{I}_A \otimes \mathcal{I}_B$ 为归一化缠结张量积态。通过固定 j 并给定和乐 $g \in \mathrm{SU}(2)$ 的边粘合 A 与 B 后，得到的边界自旋纠缠仅依赖于自旋 j ，而与和乐 g 无关:

$$E^\partial_{\mathcal{P}_g[I_A \otimes I_B]} (A \mid B) = \ln (2j + 1) . \tag{41}$$

This result is the insight that the spins of the spin network reflect the quantum correlation between spin network vertices, i.e., quanta of 3D volumes as underlined in [30-32]. It is in fact at the origin of the proposal of relating the spins to a notion of distance in loop quantum gravity [30].

该结果说明自旋网的自旋反映了自旋网顶点 (即如文献 [30-32] 强调的 3D 体积量子) 之间的量子关联。这实际上就是圈量子引力中把自旋与距离概念关联起来这一提案的起源 [30]。

An interesting feature is that the entanglement above does not depend on the holonomy between the two vertices. This is consistent with the local SU(2) gauge invariance of the theory: one can always change the SU(2) holonomy along the edge linking the two nodes A and B and in particular set $g_{AB} = \mathbb{I}$, by doing a suitable SU(2) gauge transformation at A or at B. This is actually at the heart of the "coarse-graining by gauge-fixing" procedure introduced in [33,34]. Now, if one was to consider three nodes, A, B, and C, forming a loop in the spin network graph, one could not gauge-away the SU(2) holonomy around the closed loop $A \to B \to C \to A$, and the tri-partite entanglement between the three intertwiners will depend non-trivially on this holonomy and reflect the curvature carried by the spin network [35].

一个有趣的特点是，上述纠缠不依赖于两个顶点之间的和乐。这与该理论的局域 SU(2) 规范不变性一致: 我们总能通过在节点 A 或 B 处做合适的 SU(2) 规范变换，改变连接两节点 A 与 B } 的边的 SU(2) 和乐，特别地可将其置为 $g_{AB} = \mathbb{I}$。这实际上正是文献 [33,34] 引入的“规范粗粒化”流程的核心。现在，如果考虑三个节点 A, B 和 C 在自旋网络图中构成一个闭合圈，则我们无法通过规范变换消除闭合圈 $A \to B \to C \to A$ 周围的 SU(2) 和乐，此时三个缠结之间的三部分纠缠会非平凡地依赖于该和乐，并且反映出自旋网携带的曲率 [35]。

Boundary Spins and Bulk Reconstruction

边界自旋与体重构

The boundary spin point of view for spin network is much more general than the analysis of neighboring vertices. Below, we extend the definition of the boundary spin entanglements to bounded regions of a spin network containing more than two vertices. It allows for a formulation of spin network states of quantum geometry for 3D regions with 2D boundaries and opens the door to a systematic study of the relation between bulk geometry and boundary states.

自旋网络的边界自旋视角比相邻顶点分析通用得多。下文我们将边界自旋纠缠的定义推广到包含两个以上顶点的有界自旋网络区域，它能为带二维边界的三维区域建立量子几何的自旋网络态表述，为系统研究体几何与边界态的关系打开了大门。

Indeed, the standard definition of loop quantum gravity's quantum states as wave-functions on closed graphs, as given earlier in (1), is only valid for closed 3D space manifolds, without boundaries. Now, a two-dimensional boundary in 3D space would cut links of the graph and delimit a bounded region of a spin network state, as depicted in Fig. 6. The geometry of a region with 2D boundary should thus be described with states living on open graphs.

事实上，如前文 (1) 给出的，圈量子引力量子态的标准定义是闭图上的波函数，该定义仅适用于无边界的闭三维空间流形。如图 6 所示，三维空间中的二维边界会切割图的连线，界定出自旋网络态的有界区域。因此带二维边界的区域几何应由开图上的态描述。

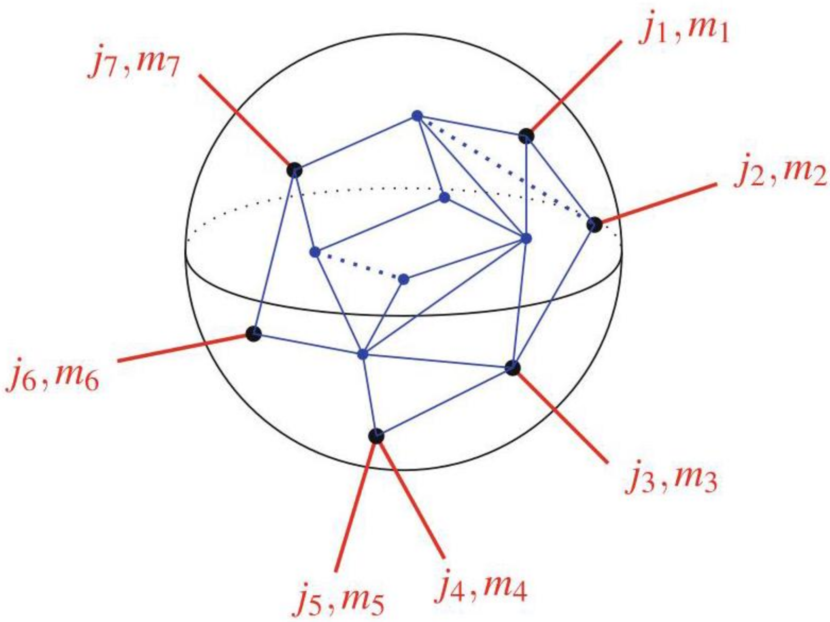

Fig. 6 Loop quantum gravity boundary data defined by the spin network puncturing the two-dimensional space boundary $\mathcal{S}_{2d} = \partial\sum_{3d}$: the interior graph γ is drawn in blue; the spin network boundary edges $e \in \partial\Gamma$ are drawn in red; they connect to the interior graph γ by the boundary vertices in black and carry boundary spin states $|j_i, m_i\rangle$, which are the quantization of the geometrical flux through the boundary surface and define quanta of area on $\mathcal{S}_{2d}$

图 6 被自旋网络穿刺的二维空间边界定义的圈量子引力边界数据 $\mathcal{S}_{2d} = \partial\sum_{3d}$: 内部图 γ 用蓝色标出；自旋网络边界边 $e \in \partial\Gamma$ 用红色标出；它们通过黑色的边界顶点连接到内部图 γ ，并携带边界自旋态 $|j_i, m_i\rangle$ ，边界自旋态是穿过边界面的几何通量的量子化，在 $\mathcal{S}_{2d}$ 上定义了面积量子

Let us call Γ an open graph. And let us distinguish the boundary $\partial\Gamma$, as the set of open edges, from the interior graph γ made from all the edges whose both ends are contained in the graph, $\gamma = \Gamma \setminus \partial\Gamma$.

我们称 Γ 为开图，并将作为开边集合的边界 $\partial\Gamma$ ，与由两端都在图内的所有边构成的内部图 γ 区分开，$\gamma = \Gamma \setminus \partial\Gamma$ 。

The boundary describes the boundary of the 3D region. Each open edge carries a spin state with an a priori arbitrary spin. This defines the boundary Hilbert space associated to a single boundary edge or in other words to a single puncture on the 2D boundary:

边界描述三维区域的边界。每条开边携带一个自旋态，其自旋先验上可以任意。这就定义了与单条边界边 (即二维边界上的单个穿刺点) 关联的边界希尔伯特空间:

$$\forall e \in \partial\Gamma,\ \mathcal{H}_e^\partial = \bigoplus_{j_e \in \frac{\mathbb{N}}{2}} V^{j_e}. \tag{42}$$

The boundary Hilbert space is then the tensor product of those single spin spaces:

边界希尔伯特空间是这些单自旋空间的张量积:

$$\mathcal{H}_{\partial\Gamma} = \bigotimes_{e\in\partial\Gamma} \mathcal{H}_e^\partial \tag{43}$$

A boundary state in $\mathcal{H}_{\partial\Gamma}$ describes the quanta of areas of the boundary surface and can be thought of as a quantum boundary condition for the bulk geometry.

$\mathcal{H}_{\partial\Gamma}$ 中的边界态描述边界面的面积量子，可看作体几何的量子边界条件。

Quantum states of the bulk geometry are now defined as gauge-covariant wave-functions of the SU(2) holonomies along the edges of the interior graph γ , valued in the boundary Hilbert space [36]:

体几何的量子态现在定义为内部图 γ 各边上 SU(2) 和乐的规范协变波函数，取值于边界希尔伯特空间 [36]:

$$\psi_\Gamma : \mathrm{SU}(2)^{E_\gamma} \to \mathcal{H}_{\partial\Gamma} \tag{44}$$

This means that, for a set of group elements $\{g_e\}_{e\in\gamma}$ on the graph, the evaluation of the wave-function $\psi_\Gamma\,[\{g_e\}]$ is a boundary state. Then, for a given boundary state $\Omega \in \mathcal{H}_{\partial\Gamma}$, the squared modulus of the scalar product in boundary space,

这意味着，对图上的一组群元 $\{g_e\}_{e\in\gamma}$ ，波函数 $\psi_\Gamma\,[\{g_e\}]$ 的求值结果就是一个边界态。那么，对给定边界态 $\Omega \in \mathcal{H}_{\partial\Gamma}$ ，边界空间中标量积的模平方

$$P_\psi^\Omega\,[\{g_e\}] = \left|\langle \Omega \mid \psi_\Gamma\,[\{g_e\}]\rangle_\partial\right|^2, \tag{45}$$

gives the probability for the bulk holonomies $\{g_e\}$ for the fixed quantum boundary condition Ω . We do not require the wave-function to be fully invariant under local SU(2) gauge transformations at each vertex of the graph, but to be SU(2)-covariant under boundary gauge transformations. This is a generic feature of boundaries in gauge field theories (see, e.g., [26]). More precisely, in our framework, we distinguish boundary vertices, to which are attached the boundary edges, from bulk vertices, which are not connected to any boundary edge. Then gauge invariance translates into

给出固定量子边界条件 Ω 下体和乐 $\{g_e\}$ 的概率。我们不要求波函数在图的每个顶点处都满足局域 SU(2) 规范变换下的完全不变性，只要求它在边界规范变换下满足 SU(2) 协变性。这是规范场论中边界的通用特征 (例如见 [26])。更准确地说，在我们的框架中，我们将连接边界边的边界顶点，与不连接任何边界边的体顶点区分开，此时规范不变性可表述为

$$\psi_\Gamma\left[\left\{h_{t(e)} g_e h_{s(e)}^{-1}\right\}\right] = \bigotimes_{e\in\partial\Gamma} h_{v(e)}^{\varepsilon_e} \psi_\Gamma\,[\{g_e\}] \in \mathcal{H}_{\partial\Gamma}, \tag{46}$$

where the vertex $v(e)$ denotes the (boundary) vertex to which the boundary edge $e \in \partial\Gamma$ is attached and where $\varepsilon_e = 1$ when the boundary edge is outgoing (i.e., $v(e) = s(e)$) and $\varepsilon_e = -1$ when the boundary edge is incoming (i.e., $v(e) = t(e)$) . When the boundary is empty, $\partial\Gamma = \varnothing$, then the boundary Hilbert space is trivial, $\mathcal{H}_\partial = \mathbb{C}$, and we recover standard wave-functions and spin networks on closed graphs.

其中顶点 $v(e)$ 表示边界边 $e \in \partial\Gamma$ 所附着的 (边界) 顶点, 当边界边为出边时 (即 $v(e) = s(e)$)), $\varepsilon_e = 1$ ；当边界边为入边时 (即 $v(e) = t(e)$)), $\varepsilon_e = -1$ 。当边界为空时, $\partial\Gamma = \varnothing$, 此时边界希尔伯特空间是平凡的, 即 $\mathcal{H}_\partial = \mathbb{C}$, 我们就得到了封闭图上的标准波函数和自旋网络。

This formulation of bulk quantum states as linear maps on the boundary Hilbert space, or in short ”boundary maps,” as introduced in [36], allows for a clearer bulk-boundary relation. For instance, it is expected that tracing out the bulk degrees of freedom, one obtains a mixed boundary state [30,37], which can be seen to lead to a bulk-induced decoherence mechanism for the boundary [38,39]. Here, the boundary state induced by the bulk state ψ_Γ is simply given by the boundary density matrix defined as

这种将体量子态表示为边界希尔伯特空间上线性映射 (简称“边界映射”) 的形式由文献 [36] 提出, 它能更清晰地给出体-边界关系。例如, 人们认为, 对体自由度求迹后会得到一个混合边界态 [30,37], 这会为边界引出体诱导退相干机制 [38,39]。这里, 由体量子态 ψ_Γ 诱导得到的边界态可直接由如下定义的边界密度矩阵给出:

$$\rho_\partial [\psi_\Gamma] = \int_{\mathrm{SU}(2)^{E_\gamma}} \prod_e \mathrm{d}g_e \, |\psi_\Gamma [\{g_e\}]\rangle\langle\psi_\Gamma [\{g_e\}]| \in \mathrm{End}\,[\mathcal{H}_{\partial\Gamma}] \,. \tag{47}$$

This is the general formulation to describe the classical and quantum correlations between boundary spins, extending the previous definition of boundary spin entanglement reviewed in Section ”Geometry as a Network of Entanglement.”

这是描述边界自旋之间经典与量子关联的通用形式, 推广了“几何即纠缠网络”一节中回顾的边界自旋纠缠的原有定义。

This density matrix actually encodes the state of the boundary once we trace out the bulk geometry and encodes all the correlations and entanglement carried by the boundary due to its embedding in the quantum geometry defined by the bulk state ψ_Γ . For instance, in the case of a black hole described in loop quantum gravity, this density matrix would contain the entanglement information between parts of a black hole horizon, as considered, for example, in [40].

当我们对体几何求迹后, 这个密度矩阵实际上就编码了边界的状态, 同时编码了边界因嵌入体量子态 ψ_Γ 定义的量子几何所携带的全部关联与纠缠。例如, 在圈量子引力描述的黑洞场景中, 这个密度矩阵会包含黑洞视界各部分之间的纠缠信息, 正如文献 [40] 中所讨论的那样。

The natural question one may ask, especially in the context of the research on the holographic principle in quantum gravity, is how much can we know on the bulk state ψ_Γ , and in particular on the graph Γ , from the boundary density matrix $\rho_\partial [\psi_\Gamma]$? As such, the bulk reconstruction problem is reformulated as the purification of the boundary state. A universal reconstruction theorem was shown in [36, 39] : whatever the density matrix on the boundary Hilbert space, one can always induce it from a spin network states based on a graph with a single loop in the bulk. The key is that the SU(2) holonomy around that loop is enough freedom to explore the whole space of boundary density matrices. Since the holonomies carried by the spin networks represent the curvature in the bulk, this theorem reflects the insight that the entanglement of the boundary geometry of the region reflects the quantum fluctuations of the curvature within the region (see [35] for further developments). However, it also shows that the boundary data does not reflect the depth of the region. Thus a truly holographic reconstruction of the bulk from the boundary crucially depends on the properties of the spin network states and requires a deeper understanding of the entanglement carried by the bulk states. Setting the foundations of the structure of entanglement in LQG is the focus of the next section.

尤其是在量子引力全息原理的研究背景下，我们很自然会提出问题: 从边界密度矩阵 $\rho_\partial[\psi_\Gamma]$ 出发，我们能获知体量子态 ψ_Γ 的多少信息，特别是图 Γ 的多少信息？据此，体重建问题可以被重新表述为边界态的纯化问题。一个通用重建定理已在 [36, 39] 中被证明: 无论边界希尔伯特空间上的密度矩阵是什么形式，它总可以由一个体中仅含单环的图对应的自旋网络态得到。核心在于，该环的 SU(2) 环绕度已经提供了足够的自由度来覆盖整个边界密度矩阵空间。由于自旋网络携带的环绕度对应体中的曲率，这个定理反映了一个深刻观点: 区域边界几何的纠缠反映了区域内部曲率的量子涨落 (进一步发展见 [35])。但它同时也表明，边界数据无法反映区域的内部深度。因此，从边界出发真正全息地重建体，关键依赖于自旋网络态的性质，还需要我们更深入地理解体态携带的纠缠。在下一节，我们将聚焦于为 LQG 中的纠缠结构建立基础。

Geometric Entanglement Entropy and the Hierarchy of States

几何纠缠熵与态层级

The kinematical Hilbert space of loop quantum gravity is spanned by spin-network states $|\Gamma, j_e, I_v\rangle$, an orthonormal basis of states that, at fixed graph Γ , simultaneously diagonalizes a complete set of commuting observables given by ultra-local operators which have the geometric interpretation of microscopic areas of each link and microscopic volume of each node of the graph. As a result, by construction, correlations at space-like separation vanish: For instance, the connected volume-volume correlation function is identically zero in each spin-network basis state. The following two questions then naturally arise: What is the behavior of correlation functions in a typical state in the Hilbert space? And how do we characterize the corner of the Hilbert space that supports states with long-range correlations as the ones of perturbative quantum fields in a fixed background spacetime? In this section we delineate recent progress in addressing these two questions using quantum-information methods.

圈量子引力的运动学希尔伯特空间由自旋网络态 $|\Gamma, j_e, I_v\rangle$ 张成，这些态构成一组标准正交基: 在给定图 Γ 上，它们同时对角化一整套由超局域算子给出的对易可观测量，这些超局域算子具有几何意义，对应图中每条边的微观面积和每个节点的微观体积。因此，按照该构造，类空间隔下的关联为零: 例如，在任意自旋网络基态中，体积-体积连通关联函数恒为零。由此自然引出两个问题: 希尔伯特空间中典型态的关联函数行为是怎样的？我们又该如何刻画希尔伯特空间中支撑长程关联态的区域——这类长程关联正是固定背景时空下微扰量子场的特征。本节我们将介绍利用量子信息方法解决这两个问题的最新进展。

Entanglement Entropy Bounds on Uncertainties and Correlations

纠缠熵对不确定性与关联的界

The notion of entanglement entropy provides a powerful tool that allows us to characterize properties of a subsystem, such as uncertainties and correlations, in terms of a single information-theoretic quantity. We describe here two results and their application to the quantum geometry of space in loop quantum gravity.

纠缠熵的概念提供了一个强大工具，让我们可以用单个信息论量表征子系统的性质，比如不确定性和关联。我们在此介绍两个结果，及其在圈量子引力的空间量子几何中的应用。

The first key result is a lower bound on the uncertainty of outcomes of measurements on a subsystem. Let us consider an isolated quantum system with bipartite Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Given a pure state $|\psi\rangle$, we can determine the probability p_k that in a measurement of an observable $\mathcal{O}_A$ in the subsystem A , we find as outcome the eigenvalue λ_k . A standard way of characterizing the probability distribution of measurement outcomes is in terms of moments of the distribution: We compute the average outcome $\langle\lambda\rangle$ and the dispersion around the average $\Delta\lambda$, and, if the probability distribution is peaked around the average, these two moments characterize well its peakedness properties. In general, however, for distributions with long tails and for multimodal distributions, average and dispersion do not provide us with a quantitative characterization of the uncertainty in the distribution of measurement outcomes. Information theoretic methods, such us the Shannon entropy of a probability distribution $S(p_k) = -\sum_k p_k \log p_k$, provide us exactly with the tool to address questions about uncertainty in these more general cases: a large Shannon entropy indicates large uncertainty in the distribution of measurement outcomes, while a zero Shannon entropy tells us that the state $|\psi\rangle$ is an eigenstate of the observable $\mathcal{O}_A$, and therefore there is a certain outcome given by the associated eigenvalue. The entanglement entropy $S_A(|\psi\rangle) = -\mathrm{Tr}_A(\rho_A \log \rho_A)$ (with $\rho_A = \mathrm{Tr}_B |\psi\rangle\langle\psi|$) provides a bound on the entropy of measurement for any set of observables in the subsystem A . Let us consider a state $|\psi\rangle$ and a complete set of commuting observables $\mathcal{O}_A^i$ (c.s.c.o.) in the subsystem A . The entropy of measurement outcomes is

第一个核心结果是子系统测量结果不确定性的下界。我们考虑一个二分希尔伯特空间为 $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ 的孤立量子系统。给定纯态 $|\psi\rangle$ ，我们可以确定概率 p_k : 测量子系统 A 中的可观测量 $\mathcal{O}_A$ 时，得到特征值 λ_k 作为结果的概率。表征测量结果概率分布的标准方法是利用分布的矩: 我们计算平均结果 $\langle\lambda\rangle$ 和平均值附近的离差 $\Delta\lambda$ ，如果概率分布集中在平均值附近，那么这两个矩就能很好地表征分布的峰度性质。但一般而言，对于长尾分布和多峰分布，平均值和离差无法定量表征测量结果分布的不确定性。信息论方法，例如概率分布 $S(p_k) = -\sum_k p_k \log p_k$ 的香农熵，恰恰为我们提供了处理这些更一般情况中不确定性问题的工具: 香农熵越大，说明测量结果分布的不确定性越大，而香农熵为零说明态 $|\psi\rangle$ 是可观测量 $\mathcal{O}_A$ 的本征态，因此测量结果一定是对应的特征值。纠缠熵 $S_A(|\psi\rangle) = -\mathrm{Tr}_A(\rho_A \log \rho_A)$ (其中 $\rho_A = \mathrm{Tr}_B |\psi\rangle\langle\psi|$) 给出了子系统 A 中任意一组可观测量测量熵的界。我们考虑子系统 A 中的一个态 $|\psi\rangle$ 和一组对易可观测量完全集 $\mathcal{O}_A^i$ (c.s.c.o.)。测量结果的熵为

$$S\left(|\psi\rangle, \mathcal{O}_A^i \to \alpha^i\right) = -\sum_{\alpha_i} p\left(|\psi\rangle, \mathcal{O}_A^i \to \alpha^i\right) \log p\left(|\psi\rangle, \mathcal{O}_A^i \to \alpha^i\right), \tag{48}$$

where the probability distribution is given by the Born rule, $p_k = \sum_\beta \left|\langle\alpha^i, \beta \mid \psi\rangle\right|^2$, and $|\alpha^i, \beta\rangle = |\alpha^i\rangle_A \otimes |\beta\rangle_B$

is an orthonormal basis of eigenstates of the c.s.c.o. $\mathcal{O}_A^i$. This quantity is bounded from below by the entanglement entropy [41]:

其中概率分布由玻恩规则给出 $p_k = \sum_\beta |\langle \alpha^i, \beta \mid \psi \rangle|^2$ ， $|\alpha^i, \beta\rangle = |\alpha^i\rangle_A \otimes |\beta\rangle_B$ 是对易可观测量完全集 $\mathcal{O}_A^i$ 本征态的正交归一基。该量由纠缠熵给出下界 [41]:

$$S_A(|\psi\rangle) \leq S\left(|\psi\rangle, \mathcal{O}_A^i \to \alpha^i\right) \; \forall \mathcal{O}_A^i \text{ c.s.c.o. in } \mathcal{H}_A . \tag{49}$$

Note that, while the right-hand side of the inequality depends both on the state and on the observables, the entanglement entropy on the left-hand side depends only on the state and on the subsystem A that those observables probe. Therefore the entanglement entropy provides a universal lower bound on uncertainties for observables in a subsystem. In the next section, we use this result to put bounds on the uncertainty of geometric observables in loop quantum gravity.

注意，虽然不等式的右手边同时依赖于态和可观测量，左手边的纠缠熵仅依赖于态和这些可观测量探测的子系统 A 。因此纠缠熵给出了子系统中可观测量不确定性的通用下界。在下一节中，我们将利用该结果给出圈量子引力中几何可观测量不确定性的界。

The second key result is an upper bound on correlations of observables. Let us consider an isolated quantum system with tripartite Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$. Given a pure state $|\psi\rangle$, we can compute the connected correlation function $\mathcal{G}$ of two bounded observables $\mathcal{O}_A$ and $\mathcal{O}_B$ acting, respectively, on subsystems A and B :

第二个关键结论是可观测量关联的上界。我们考虑一个三分希尔伯特空间 $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ 的孤立量子系统。给定纯态 $|\psi\rangle$ ，我们可以计算分别作用在子系统 A 和 B 上的两个有界可观测量 $\mathcal{O}_A$ 和 $\mathcal{O}_B$ 的连通关联函数 $\mathcal{G}$:

$$\mathcal{G} = \langle \psi | \mathcal{O}_A \mathcal{O}_B | \psi \rangle - \langle \psi | \mathcal{O}_A | \psi \rangle \langle \psi | \mathcal{O}_B | \psi \rangle . \tag{50}$$

The mutual information $S_{AB|C}(|\psi\rangle)$ of the subsystems A and B can be expressed in terms of the entanglement entropies of A, B , and C and is given by

子系统 A 和 B 的互信息 $S_{AB|C}(|\psi\rangle)$ 可以用 A, B 和 C 的纠缠熵来表示，其表达式为

$$S_{AB|C}(|\psi\rangle) = S_A(|\psi\rangle) + S_B(|\psi\rangle) - S_C(|\psi\rangle) . \tag{51}$$

Remarkably, the mutual information provides us with an upper bound on the correlation function $\mathcal{G}$,[41,42]:

值得注意的是，互信息给出了关联函数 $\mathcal{G}$ 的上界 [41,42]:

$$\frac{\left(\langle \psi | \mathcal{O}_A \mathcal{O}_B | \psi \rangle - \langle \psi | \mathcal{O}_A | \psi \rangle \langle \psi | \mathcal{O}_B | \psi \rangle\right)^2}{2\left\|\mathcal{O}_A\right\|^2 \left\|\mathcal{O}_B\right\|^2} \leq S_{AB|C}(|\psi\rangle) . \tag{52}$$

Note that, while the left-hand side of the inequality depends on the observables $\mathcal{O}_A$ and $\mathcal{O}_B$, the right-hand side depends only on the subspaces they act on. Therefore the mutual information provides a universal upper bound on correlations. In the next section, we use this result to put bounds on correlations of geometric observables in loop quantum gravity.

注意，该不等式的左端依赖于可观测量 $\mathcal{O}_A$ 和 $\mathcal{O}_B$ ，而右端仅依赖于它们作用的子空间。因此互信息为关联给出了通用上界。在下一节中，我们将利用这一结论为圈量子引力中几何可观测量的关联给出边界。

Throughout the analysis we assume that the system is in a pure state $|\psi\rangle$. As a result, the entropy in the complement $\mathcal{H}_C$ equals the entropy in the factor $\mathcal{H}_A \otimes \mathcal{H}_B$ probed by the set of observables $\mathcal{O}_A$ and $\mathcal{O}_B$, i.e., $S_C(|\psi\rangle) = S_{AB}(|\psi\rangle)$ in (51). We note that the assumption that the system is in a pure state is not a restriction: We adopt the point of view that, if the quantum geometry of space was in a mixed state, then we simply have to take into account a larger Hilbert space that includes the purifying degrees of freedom. For instance, in the presence of matter, the complement $\mathcal{H}_C$ describes both the quantum geometry not probed by the observables $\mathcal{O}_A$ and $\mathcal{O}_B$ and the matter degrees of freedom.

在整个分析过程中，我们假设系统处于纯态 $|\psi\rangle$ 。因此，补集 $\mathcal{H}_C$ 的熵等于可观测量集合 $\mathcal{O}_A$ 和 $\mathcal{O}_B$ 探测的因子 $\mathcal{H}_A \otimes \mathcal{H}_B$ 的熵，即式 (51) 中的 $S_C(|\psi\rangle) = S_{AB}(|\psi\rangle)$ 。我们要说明，系统处于纯态的假设并非限制: 我们的观点是，如果空间的量子几何处于混合态，我们只需引入更大的、包含纯化自由度的希尔伯特空间即可。例如，存在物质时，补集 $\mathcal{H}_C$ 同时描述未被可观测量 $\mathcal{O}_A$ 和 $\mathcal{O}_B$ 探测的量子几何与物质自由度。

Hierarchy of States: Volume-Law, Area-Law, and Zero-Law States

态层级: 体积律、面积律与零律态

In loop quantum gravity, the quantum geometry of space is described by a state $|\psi\rangle$ in the kinematical Hilbert space $\mathcal{H}^{\text{kin}}$. A semiclassical state is peaked on a specific value of the intrinsic and the extrinsic geometry of space. This condition fixes only the expectation value and dispersion of local geometric operators but leaves correlations at space-like separations unspecified. On the other hand, in perturbative quantum field theory on a given classical background geometry, these correlations take a specific long-range form. It becomes crucial then to identify the class of semiclassical states $|\psi\rangle$ that capture the perturbative effective-field-theory regime of the theory, by prescribing also correlation functions. Quantum information methods, and in particular the results (49) and (52), provide a new perspective on how to restrict the class of semiclassical states by moving the focus from correlation functions to a hierarchy of scaling laws for the entanglement entropy of a region: volume-law, area-law, and zero-law states. As we will see, area-law states provide a condition for semiclassicality.

在圈量子引力中，空间的量子几何由运动学希尔伯特空间 $\mathcal{H}^{\text{kin}}$ 中的态 $|\psi\rangle$ 描述。半经典态集中在空间内禀几何与外禀几何的特定取值上。该条件仅固定了局域几何算符的期望值和色散，却未限定类空间隔的关联。另一方面，在给定经典背景几何上的微扰量子场论中，这些关联具有特定长程形式。因此，关键在于通过额外规定关联函数，识别出能够描述理论微扰有效场论区域的半经典态类 $|\psi\rangle$。量子信息方法，尤其是结果 (49) 和 (52)，提供了全新视角: 通过将研究重心从关联函数转移到区域纠缠熵的标度律层级，以此约束半经典态类，即体积律态、面积律态与零律态。我们将会看到，面积律态给出了半经典性的判定条件。

The kinematical Hilbert space $\mathcal{H}^{\text{kin}}$ is spanned by spin-network states $|\Gamma, j_e, I_v\rangle$ with graph Γ, spin j_e associated to its edges, and intertwiners I_v associated to its vertices. To illustrate the main ingredients of the hierarchy of scaling laws, we consider a specific sector of $\mathcal{H}^{\text{kin}}$: We fix both the graph Γ and the spins j_e. As a result, we obtain the sector

运动学希尔伯特空间 $\mathcal{H}^{\text{kin}}$ 由自旋网络态 $|\Gamma, j_e, I_v\rangle$ 张成，这些态对应图 Γ，边带有自旋 j_e，顶点带 intertwiners(缠绕算符) I_v。为说明标度律层级的核心要素，我们考虑 $\mathcal{H}^{\text{kin}}$ 的一个特定区域: 同时固定图 Γ 和自旋 j_e，最终得到该区域

$$\mathcal{H}^{\text{kin}}_{\Gamma, j_\ell} = \bigotimes_{v\in\Gamma} \mathcal{I}_v(j_e) \tag{53}$$

where $\mathcal{I}_v(j_e)$ is the intertwiner Hilbert space at the vertex v. Having fixed the graph Γ, we have an immediate notion of connectivity and of geometric regions. Moreover, at fixed spins, the Hilbert space reduces to a tensor product over finite dimensional Hilbert spaces $\mathcal{I}_v$, one per node. This is the same structure present in ordinary many-body quantum system, and we can easily bring in methods and perspectives from condensed matter theory. We discuss later how to generalize these methods to a sum over spins and graphs.

其中 $\mathcal{I}_v(j_e)$ 是顶点 v 处的 intertwiner(缠绕算符) 希尔伯特空间。固定图 Γ 后，我们可以直接定义连通性与几何区域。此外，自旋固定后，希尔伯特空间可约化为每个节点对应一个有限维希尔伯特空间 $\mathcal{I}_v$ 的张量积。该结构与普通多体量子系统的结构一致，因此我们可以直接引入凝聚态理论的方法和视角，我们后文会讨论如何将这些方法推广到自旋与图的求和中。

For concreteness, we restrict attention to a regular graph: a cubic lattice Γ with a finite number N of sites (or vertices) and periodic boundary conditions (torus topology). We assume also that the spins associated to each edge of the graph are fixed and equal to j_0. As a result, we have a many-body quantum system, where each body is a quantum polyhedron [22] with six faces of equal area $a_0 = 8\pi G\hbar\gamma\sqrt{j_0(j_0+1)}$. We can write the generic state as

为具体起见，我们将注意力限制在一个规则图上: 一个具有有限数量 N 个节点 (或顶点) 和周期性边界条件 (环面拓扑) 的立方晶格 Γ。我们还假设与图的每条边相关联的自旋是固定的，且都等于 j_0。因此，我们得到了一个多体量子系统，其中每个体都是一个具有六个面积相等的面 $a_0 = 8\pi G\hbar\gamma\sqrt{j_0(j_0+1)}$ 的量子多面体 [22]。我们可以将一般状态写为

$$|\psi\rangle = \sum_{i_1,\ldots,i_N} \psi_{i_1,\ldots,i_N} |i_1\rangle \otimes \cdots \otimes |i_N\rangle, \tag{54}$$

with $|i_n\rangle \in \mathcal{I}_v = \mathrm{Inv}_{SU(2)}\left[\left(V^{j_0}\right)^{\otimes 6}\right]$ an orthornormal basis of intertwiners at each vertex. The factorized structure (53) allows us to identify an immediate notion of geometric regions containing N_A contiguous vertices and geometric entanglement entropy associated to the subsystem decomposition $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. We are now ready to describe a hierarchy of states with different properties of the quantum geometry.

其中 $|i_n\rangle \in \mathcal{I}_v = \mathrm{Inv}_{SU(2)}\left[\left(V^{j_0}\right)^{\otimes 6}\right]$ 是每个顶点处 intertwiners(缠绕算符) 的标准正交基。分解结构 (53) 允许我们直接定义包含 N_A 个相邻顶点的几何区域，以及对应子系统分解 $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ 的几何纠缠熵。我们现在可以描述具有不同量子几何性质的态层级了。

Zero-Law States Spin-network basis states $|\Gamma, j_0, i_n\rangle = |i_1\rangle \otimes \cdots \otimes |i_N\rangle$ are product states over the intertwiners associated to the vertices of the cubic lattice. As a result, if we consider a region A of the lattice, the entanglement entropy S_A vanishes: they are zero-law states. To illustrate their properties, let us consider some examples. We can take $|i_n\rangle$ to be given by eigenstates of the oriented volume operator V_n . In this case, dihedral angles $\theta_{ne\tilde{e}}$ will have large uncertainties (with a non-zero probability of finding both cuboids and pentagonal wedges), but the outcomes of measurements of the volume have zero measurement entropy, $S(|\psi\rangle, V_n \to v_n) = 0$. Therefore they saturate the uncertainty bound (49). We can also consider coherent states for the intrinsic geometry of a cubic lattice by choosing the state (54) with $\psi_{i_1,\ldots,i_N} = \Phi_{i_1} \cdots \Phi_{i_N}$ and Φ_{i_n} coherent intertwiners peaked over the Euclidean geometry of a cube [22-24]. In this case, the average volume is the one of a cube with faces of fixed area, the average dihedral angles are right angles, but there are fluctuations around the average. Are fluctuations at nearby nodes correlated? The correlation bound (52) tells us that all connected correlation functions, such as angle-angle correlations $\mathcal{G} = \langle \theta_{ne\tilde{e}} \theta_{n'e'\tilde{e}'} \rangle - \langle \theta_{ne\tilde{e}} \rangle \langle \theta_{n'e'\tilde{e}'} \rangle = 0$, vanish because the mutual information between regions is zero. Therefore there are no correlations at space-like separation for zero-law states.

零定律态自旋网基态 $|\Gamma, j_0, i_n\rangle = |i_1\rangle \otimes \cdots \otimes |i_N\rangle$ 是立方晶格顶点对应 intertwiners 上的乘积态。因此，若我们考虑晶格的一个区域 A ，纠缠熵 S_A 为零: 它们就是零定律态。我们通过几个例子说明其性质。我们可以取 $|i_n\rangle$ 为定向体积算符 V_n 的本征态。在这种情况下，二面角 $\theta_{ne\tilde{e}}$ 会存在很大不确定性 (找到长方体和五边形楔的概率均非零)，但体积测量结果的测量熵为零，$S(|\psi\rangle, V_n \to v_n) = 0$ 。因此它们满足不确定界 (49) 的饱和情况。我们还可以通过选择态 (54)，令其中 $\psi_{i_1,\ldots,i_N} = \Phi_{i_1} \cdots \Phi_{i_N}$ 和 Φ_{i_n} 为峰在立方体欧几里得几何上的相干 intertwiner，得到立方本征内禀几何的相干态 [22-24]。在这种情况下，平均体积等于面面积固定的立方体的体积，平均二面角为直角，但平均值周围存在涨落。相邻节点的涨落存在关联吗？关联界 (52) 告诉我们，由于区域间互信息为零，所有连通关联函数 (例如角-角关联 $\mathcal{G} = \langle \theta_{ne\tilde{e}} \theta_{n'e'\tilde{e}'} \rangle - \langle \theta_{ne\tilde{e}} \rangle \langle \theta_{n'e'\tilde{e}'} \rangle = 0$) 都等于零。因此零定律态在类空间隔下不存在任何关联。

Volume-Law States The Hilbert space of a cubic lattice at fixed spin j_0 has finite dimension $\dim \mathcal{H}_{\Gamma, j_0} = (\dim \mathcal{I}_n)^N$ where N is the number of nodes in the lattice and $\dim \mathcal{I}_n$ is the dimension of each intertwiner space. For a finite dimensional Hilbert space, there is a notion of random state $|\psi\rangle$, that is, a state extracted randomly from the ensemble of normalized vectors with uniform probability distribution. Equivalently, one can consider a reference state $|\psi_0\rangle$ and a random state $|\psi\rangle = U|\psi_0\rangle$, where U is a random unitary extracted from the ensemble | of unitary matrices distributed according to the Haar measure. Remarkably, the entanglement entropy of random states has typicality properties: For a random state, the probability of finding a given value of the entanglement entropy is peaked at an average value $\langle S_A \rangle$ with a small dispersion ΔS_A [43-45]. Let us consider a region A that contains a finite fraction of the nodes of the cubic lattice and discuss the limit of large region $N_A \to \infty$ and large lattice, $N \to \infty$, with fixed finite ratio f_A :

体积定律态固定自旋 j_0 下立方晶格的希尔伯特空间维度有限，为 $\dim \mathcal{H}_{\Gamma,j_0} = (\dim \mathcal{I}_n)^N$，其中 N 是晶格的节点数，$\dim \mathcal{I}_n$ 是每个 intertwiner 空间的维度。对于有限维希尔伯特空间，存在随机态 $|\psi\rangle$ 的概念，即从归一化向量的均匀概率分布系综中随机抽取的态。等价地，可以考虑参考态 $|\psi_0\rangle$ 和由满足哈尔测度分布的幺正矩阵生成的随机态 $|\psi\rangle = U\,|\psi_0\rangle$, where U is a random unitary extracted from the ensemble | 。值得注意的是，随机态的纠缠熵具有典型性: 对于随机态，得到某一特定纠缠熵值的概率峰在平均值 $\langle S_A\rangle$ 处，且色散 ΔS_A 很小 [43-45]。现在考虑一个包含立方晶格有限比例节点的区域 A，讨论大区域 $N_A \to \infty$、大晶格、$N \to \infty$ 且比例 f_A 固定有限的极限:

$$f_A = \frac{N_A}{N} \le \frac{1}{2}. \tag{55}$$

In a random state, the average entanglement entropy of the region A is

在随机态中，区域 A 的平均纠缠熵为

$$\langle S_A\rangle = f_A V \frac{\log d_n}{v_0} - \frac{1}{2} {d_n}^{-(1-2f_A)V/v_0} + O(d_n)\,, \tag{56}$$

where v_0 is the average volume of each node. The number $d_n = \dim \mathcal{I}_n$ is the dimension of each six-valent intertwiner space. It is equal to $d_n = 5$ for $j_0 = 1/2$ and scales as $d_n \sim \frac{8}{\pi} {j_0}^3$ for $j_0 \gg 1$. As the formula shows, at the leading order, the entanglement entropy of the region A scales linearly with the volume $V_A = f_A V$ of the region. While this is the average entanglement entropy of random states, its value is also typical: the dispersion around the average is exponentially small in the volume,

其中 v_0 是每个节点的平均体积。$d_n = \dim \mathcal{I}_n$ 是每个六价缠结空间的维数。当 $j_0 = 1/2$ 时它等于 $d_n = 5$，当 $j_0 \gg 1$ 时它标度为 $d_n \sim \frac{8}{\pi} {j_0}^3$。如公式所示，领头阶下，区域 A 的纠缠熵与该区域的体积 $V_A = f_A V$ 呈线性标度关系。这是随机态的平均纠缠熵，同时该取值也具有典型性: 平均值附近的色散随体积呈指数减小，

$$\Delta S_A = \sqrt{\frac{1}{2} - \frac{1}{4}\delta_{f,1/2}}\, {d_n}^{-(1-f_A)V/v_0} + O(d_n)\,. \tag{57}$$

The uncertainty bound (49) tells us that, because of the typical volume law for a random state, there is no observable in the region A that has vanishing uncertainty. In fact for a set of observables probing a region of average volume V_A , i.e., a c.s.c.o. in $\mathcal{H}_A$, the measurement entropy (48) scales linearly with the volume. Moreover, one can consider the mutual information for two regions A and B . The leading order volume-law terms cancel in (51), and, as a result, the typical mutual information in a random state is exponentially small in the total volume V . We can use then the correlations bound (52) to conclude that, for instance, angle-angle correlations in a random state scale as

不确定性界 (49) 表明，由于随机态满足典型体积律，区域 A 中不存在不确定性消失的可观测量。事实上，对于一组探测平均体积 V_A 区域的可观测量，即 $\mathcal{H}_A$ 中的完备对易可观测量集，测量熵 (48) 与体积呈线性标度。此外，我们可以考虑两个区域 A 和 B 之间的互信息。领头阶体积律项会在 (51) 中抵消，因此随机态中的典型互信息随总体积 V 呈指数减小。我们随后可以利用关联界 (52) 得到结论，例如，随机态中的角度-角度关联标度为

$$\mathcal{G} = \langle \theta_{ne\tilde{e}} \theta_{n'e'\tilde{e}'} \rangle - \langle \theta_{ne\tilde{e}} \rangle \langle \theta_{n'e'\tilde{e}'} \rangle = O(d_n), \tag{58}$$

and in general, correlation functions between any two space-like separated observ-ables are exponentially small in a random state.

一般而言，任意两个类空分离可观测量的关联函数在随机态中都呈指数减小。

Area-Law States In many-body quantum systems, area-law states arise as low-energy states of Hamiltonians with local interactions. To lower the average energy, the state needs to have a long-range decay of correlation functions which result in geometric entanglement entropy that scales with the area of the boundary of the region [46-48]. This behavior is to be contrasted to the one of high-energy states which typically satisfy a volume law. In fact, random states of fixed energy (away from the edge of the energy spectrum) show a thermal behavior for subsystems (known as eigenstate thermalization hypothesis) [49-51] and result in a typical volume-law entanglement entropy with scaling coefficient dependent on the energy [44,45]. Another relevant example of states that do not belong to the low-energy spectrum is zero-law states, which by definition are product states: the lack of space-like correlations results in a contribution to the spatial coupling in the Hamiltonian which makes them high-energy states. In quantum field theory, these zero-law states do not even belong to the Fock space of the theory as they are not Hadamard states [52,53] and, if an ultraviolet cutoff is introduced, they can be understood as states with divergent energy density. To better characterize the class of area-law states in a way that is insensitive to the ultraviolet behavior of the theory, it is useful to consider the mutual information $S_{AB|C}(|\psi\rangle)$ between a region A and the complement B of the enlarged region AC , where C is a "safety corridor" around A [54]. In loop quantum gravity, area-law states at fixed spins arise from long-range intertwiner correlations. Their geometric interpretation is discussed in the next section.

面积定律态在多体量子系统中，面积定律态作为具有局域相互作用的哈密顿量的低能态出现。为了降低平均能量，该态需要具有关联函数的长程衰减，这会导致几何纠缠熵与区域边界的面积成比例 [46 - 48]。这种行为与通常满足体积定律的高能态的行为形成对比。事实上，固定能量 (远离能谱边缘) 的随机态对于子系统表现出热行为 (即本征态热化假设)[49 - 51]，并导致典型的体积定律纠缠熵，其缩放系数取决于能量 [44,45]。另一个不属于低能谱的态的相关例子是零定律态，根据定义，它们是直积态: 类空间相关性的缺乏导致哈密顿量中空间耦合的贡献，使它们成为高能态。在量子场论中，这些零定律态甚至不属于该理论的福克空间，因为它们不是哈达玛态 [52,53]，并且，如果引入紫外截断，它们可以被理解为能量密度发散的态。为了以一种对理论的紫外行为不敏感的方式更好地表征面积定律态的类别，考虑区域 A 与扩大区域 AC 的补集 B 之间的互信息 $S_{AB|C}(|\psi\rangle)$ 是有用的，其中 C 是 A 周围的“安全走廊”[54]。在圈量子引力中，固定自旋的面积定律态源于长程缠结算符相关性。下一节将讨论它们的几何解释。

We note here that the conjectured relation between entanglement in loop quantum gravity and entanglement in quantum field theory relies on the same key assumption used in the investigation of the graviton propagator in spinfoams [55, 56]. In quantum field theory, there is a fixed background geometry that allows us to choose a region and then compute entanglement of quantum fields. In loop quantum gravity, when one considers a semiclassical state, the expectation value of the geometry determines an effective background, and the quantum fluctuations of the geometry reproduce the correlations of the metric perturbations in the effective field theory. It is this matching of correlation functions that results in a matching of entanglement entropies in the semiclassical regime.

我们在此指出，圈量子引力中的纠缠与量子场论中的纠缠之间的猜想关系，依赖于自旋泡沫引力子传播子研究中使用的同一个关键假设 [55, 56]。在量子场论中，存在固定背景几何，允许我们选定区域后计算量子场的纠缠。在圈量子引力中，当我们考虑半经典态时，几何的期望值确定了一个有效背景，几何的量子涨落会重现有效场论中度规扰动的关联。正是这种关联函数的匹配，使得半经典 regime 中的纠缠熵也相互匹配。

Area-Law States and the Architecture of Spacetime Geometry

面积定律态与时空几何的结构

In a non-perturbative theory of quantum gravity, low-energy-density arguments (as the ones discussed above for many-body systems and quantum field theory) are problematic as they assume the existence of a classical background geometry with respect to which the energy density is defined. It is then useful to reverse the logic. The architecture conjecture [57] uses quantum-information methods as a probe of semiclassicality of physical states in quantum gravity:

在非微扰量子引力理论中，低能密度论证 (就如上文针对多体系统和量子场论讨论的这类论证) 存在问题，因为这类论证假定了定义能量密度所依赖的经典背景几何是存在的。因此反过来推导逻辑会更有效。结构猜想 [57] 利用量子信息方法探究量子引力中物理态的半经典性质:

Entanglement and the architecture of space-time. In a theory of quantum gravity, for any sufficiently large causal domain in a semiclassical space-time, the entanglement entropy between the degrees of freedom describing a given causal domain R and those describing its complement is finite and, to leading order, takes the area-law form $S_R(|\psi\rangle) = 2\pi\,\mathrm{Area}(\partial R)/L_P^2 + \dots$, where $|\psi\rangle$ is the quantum state of the semiclassical space-time geometry and $\mathrm{Area}(\partial R)$ is the expectation value of the area of the 2-dimensional corner of the causal domain R .

纠缠与时空结构。在量子引力理论中，对于半经典时空内任意足够大的因果域，描述给定因果域 R 的自由度与描述其补集的自由度之间的纠缠熵是有限的，且在领头阶满足面积定律形式 $S_R(|\psi\rangle) = 2\pi\,\mathrm{Area}(\partial R)/L_P^2 + \dots$，其中 $|\psi\rangle$ 是半经典时空几何的量子态，$\mathrm{Area}(\partial R)$ 是因果域 R 二维边界面积的期望值。

The conjecture is motivated by the area-law behavior of the vacuum state in many-body systems with local interactions and by the vacuum entanglement entropy in quantum fields theory on Minkowski and on curved spacetimes. In quantum field theory, the vacuum entanglement entropy is ultraviolet divergent and scales with the area only if an ultraviolet cutoff is introduced, either by putting the theory on a lattice [46,47], or by introducing a safety corridor C [54], or by defining the region using a coarse-grained sub-algebra of observables [58]. On the other hand, in quantum gravity the finite Planck area coefficient $L_P^2/2\pi$ is expected, as supported by various lines of evidence from the renormalization of the gravitational coupling in black hole backgrounds [59], from the thermodynamics of space-time geometry [60], from the holographic entanglement entropy [61], and from the entanglement entropy of the Rindler horizon in perturbative quantum gravity [62].

该猜想的动机来自: 具有局域相互作用的多体系统真空态的面积定律行为，以及闵氏时空和弯曲时空上量子场论的真空纠缠熵。量子场论中，真空纠缠熵是紫外发散的，只有引入紫外截断后才会满足面积标度——无论是将理论格点化 [46,47]、引入安全走廊 C [54]，还是通过粗粒化可观测量子代数定义区域 [58]。另一方面，量子引力中预期会出现有限的普朗克面积系数 $L_P^2/2\pi$，这一点得到多方面证据的支持: 黑洞背景下引力耦合的重整化 [59]、时空几何热力学 [60]、全息纠缠熵 [61]，以及微扰量子引力中林德勒视界的纠缠熵 [62]。

Often, it is useful to characterize states in the Hilbert space independently of the Hamiltonian and directly in terms of properties of the correlation functions. This procedure identifies a corner of the Hilbert space that can then be used as a variational ansatz for the dynamics. In many-body quantum systems, this approach has a computational advantage over exact diagonalization of the Hamiltonian. Using a similar strategy in loop quantum gravity, where a local notion of energy is not available, allows us to identify classes of states with a desired scaling of the correlation functions before addressing the difficult problem of the dynamics. A general technique for addressing this problem is provided by squeezed vacua for spin-networks, a family of states $|\Gamma, \gamma\rangle$ that spans the Hilbert space of loop quantum gravity, which are labeled by a graph Γ and a matrix γ_{ij}^{AB} which encodes quantum correlations [63-65]. Squeezed vacua provide an overcomplete basis of loop quantum gravity that is tailored to the study of the entanglement structure of space in a semiclassical spacetime geometry with perturbative quantum fluctuations. The definition of squeezed vacua for loop quantum gravity involves three key ingredients. The first is the use of a bosonic Hilbert space as done in the spinor representation of loop quantum gravity [66-69]. The idea is based on Schwinger's oscillator model of spin [70]: Given two oscillators with creation operators $a_1^\dagger, a_2^\dagger$ and vacuum $|0\rangle$, the state of definite spin is given by $|j, m\rangle = \frac{\left(a_1^\dagger\right)^{j+m}}{\sqrt{(j+m)!}} \frac{\left(a_2^\dagger\right)^{j-m}}{\sqrt{(j-m)!}} |0\rangle$. The second ingredient is the construction of an overcomplete basis of squeezed vacua for a bosonic lattice introduced in [65]. In the simple case of a single harmonic oscillator with vacuum $|0\rangle$, a squeezed vacuum is given by $|\gamma\rangle = e^{\frac{1}{2}\gamma a^\dagger a^\dagger} |0\rangle$, where γ is a complex number. The third ingredient is an improvement of the original loop expansion that is at the roots of loop quantum gravity [71]. Using bosonic variables, one can introduce normal-ordered Wilson loops and use the loop expansion to define a projector from bosonic states to loop states [63]. The result of this construction is a formulation of squeezed vacua for loop quantum gravity, written as a superposition of spin-network basis states $|\Gamma, j_e, I_v\rangle$ (Eq. 102 in [63]):

通常, 我们可以独立于哈密顿量, 直接利用关联函数的性质来刻画希尔伯特空间中的态, 这一方法十分实用。通过该步骤我们可以确定希尔伯特空间的一个区域, 后续可将其用作动力学的变分拟设。在多体量子系统中，这种方法比哈密顿量的精确对角化更具计算优势。圈量子引力中不存在局域的能量概念, 在该理论中采用类似策略, 我们可以在处理复杂的动力学问题之前, 先筛选出关联函数满足预期标度性质的态族。针对该问题的一般技术由自旋网络压缩真空提供,这是张成圈量子引力希尔伯特空间的一个态族 $|\Gamma, \gamma\rangle$, 由图 Γ 和编码量子关联的矩阵 γ_{ij}^{AB} 标记 [63-65]。压缩真空为圈量子引力提供了一组过完备基, 专门用于研究半经典时空几何中空间的纠缠结构, 其中包含微扰量子涨落。圈量子引力压缩真空的定义包含三个核心要素: 第一是如同圈量子引力的旋量表示那样，引入玻色希尔伯特空间 [66-69]。这一想法基于施温格自旋振荡器模型 [70]: 给定两个产生算符为 $a_1^\dagger, a_2^\dagger$ 的振荡器, 真空为 $|0\rangle$, the state of definite spin is given by $|j, m\rangle = \frac{(a_1^\dagger)^{j+m}}{\sqrt{(j+m)!}} \frac{(a_2^\dagger)^{j-m}}{\sqrt{(j-m)!}} |0\rangle$ 。第二个要素是文献 [65] 中提出的玻色晶格过完备压缩真空基的构造。对于真空为 $|0\rangle$, a squeezed vacuum is given by $|\gamma\rangle = e^{\frac{1}{2}\gamma a^\dagger a^\dagger}|0\rangle$ 的单个简谐振子简单情形，其中 γ 为复数。第三个要素是对作为圈量子引力根基的原始圈展开的改进 [71]。利用玻色变量，我们可以引入正规序威尔逊圈，并通过圈展开定义从玻色态到圈态的投影算符 [63]。该构造最终给出了圈量子引力压缩真空的表述，它可以写为自旋网络基态 $|\Gamma, j_e, I_v\rangle$ 的叠加 (文献 [63] 中式 (102)):

$$\left|\Gamma, \gamma_{ij}^{AB}\right\rangle = \sum_{j_e, I_v} c_{j_e, I_v}\left(\gamma_{ij}^{AB}\right) |\Gamma, j_e, I_v\rangle \tag{59}$$

where the expansion coefficients $c_{j_e, i_n}\left(\gamma_{ij}^{AB}\right) \equiv \left\langle \Gamma, j_e, I_v \mid \Gamma, \gamma_{ij}^{AB} \right\rangle$ are expressed as Gaussian integrals

其中展开系数 $c_{j_e, i_n}\left(\gamma_{ij}^{AB}\right) \equiv \left\langle \Gamma, j_e, I_v \mid \Gamma, \gamma_{ij}^{AB} \right\rangle$ 可以表示为高斯积分

$$c_{j_e, I_v}\left(\gamma_{ij}^{AB}\right) = \int \frac{d^{4E} z d^{4E} \bar{z}}{\pi^{4E}} Z_{j_e, I_v} \mathrm{e}^{-z_i^A \bar{z}_A^i + \frac{1}{2}\gamma_{ij}^{AB} \bar{z}_A^i \bar{z}_B^j}, \tag{60}$$

with the polynomial insertion

带有多项式插入项

$$Z_{j_e, I_v} = \sum_{m_i=-j_i}^{+j_i} \left(\prod_{v=1}^{N} [\bar{I}_v]_{m_1 \cdots m_{|v|}}\right) \left(\prod_{i=1}^{2E} \frac{\left(z_i^0\right)^{j_i - m_i}}{\sqrt{(j_i - m_i)!}} \frac{\left(z_i^1\right)^{j_i + m_i}}{\sqrt{(j_i + m_i)!}}\right). \tag{61}$$

The advantage of using squeezed vacua is that they encode efficiently properties of a physically relevant corner of the Hilbert space into a matrix γ_{ij}^{AB} , which determines the correlations at space-like separation that can be bound using formula (52). Note that the matrix γ_{ij}^{AB} appears only in the exponent in (60), and it couples the spinorial variables z_i^A attached to the endpoints of the links of the graph Γ . We note also that squeezed vacua automatically induce a superposition over spins with specific coefficients which include also $j_e = 0$ and therefore a sum over all subgraphs. Information-theoretic methods allow us to characterize quantum states of the spacetime geometry in terms of their entanglement structure and mutual information. In particular, for $\gamma_{ij}^{AB} = 0$, the state reduces to the Ashtekar-Lewandowski vacuum (projected to the graph Γ), and it satisfies a zero-law for the geometric entanglement entropy. A non-trivial area law arises for Bell-network states [72, 73] which corresponds to a short-ranged squeezing matrix $\gamma_{ij}^{AB} = \lambda_{e(i,j)} \varepsilon^{AB}$ whose only non-vanishing components are the edges $e(i, j)$ of the graph Γ . These states provide the first concrete realization of the conjecture on entanglement and the architecture of spacetime put forward in [57] as they show how a semiclassical geometry arises by gluing nearby quantum polyhedra with entanglement [72].

使用压缩真空的优势在于，它能将希尔伯特空间中一个物理相关区域的性质高效编码到矩阵 γ_{ij}^{AB} 中，该矩阵决定了类空间隔下的关联，我们可通过公式 (52) 约束这类关联。注意矩阵 γ_{ij}^{AB} 仅出现在式 (60) 的指数项中，它耦合了连接图 Γ 各链端点的旋量变量 z_i^A 。我们还发现，压缩真空会自动诱导出具有特定系数的自旋叠加，系数中也包含 $j_e = 0$ ，因此相当于对所有子图求和。信息论方法允许我们依据纠缠结构与互信息刻画时空几何量子态的特征。尤其当满足 $\gamma_{ij}^{AB} = 0$ 时，该态约化为阿西卡-勒万多夫斯基真空 (投影到图 Γ 上)，且满足几何纠缠熵的零律。对于贝尔网络态 [72, 73] 会产生非平凡面积律，这类态对应短程压缩矩阵 $\gamma_{ij}^{AB} = \lambda_{e(i,j)}\varepsilon^{AB}$ ，其仅有的非零分量位于图 Γ 的边 $e(i,j)$ 上。这些态首次具体实现了文献 [57] 提出的关于纠缠与时空结构的猜想: 它们展示了半经典几何如何通过纠缠粘接相邻量子多面体产生 [72]。

In order to distinguish between a microscopic area law and an effective area law, it is useful to introduce the notion of geometric entanglement entropy with a ”safety corridor” around the region [54]: one considers a region A containing N_A nodes of the graph, an enlargement AC that surrounds the region with a corridor with N_C nodes, and the complement B . The quantity of interest is then the geometric mutual information (51) in the limit $N_A \to \infty, N_C \to \infty$, with $N_C/N_A \to 0$. In this limit, states that have only short-range correlations as Bell-network states satisfy a zero-law for the mutual information. On the other hand, when the squeezing matrix has also long-range non-vanishing components, for instance, with an inverse-distance decay with respect to the average geometry [64], one finds an effective area law for the mutual information $S_{AB|C}(|\Gamma,\gamma\rangle)$ of squeezed vacua.

为了区分微观面积律与有效面积律, 引入带“安全通道”的几何纠缠熵概念是很有用的, 即在研究区域外预留一圈安全通道 [54]: 我们考虑一个包含图中 N_A 个节点的区域 A , 区域的扩大部分 AC 围绕原区域形成一个带有 N_C 个节点的通道,以及补区域 B 。我们关注的物理量是极限 $N_A \to \infty, N_C \to \infty$ 下 (满足 $N_C/N_A \to 0$) 的几何互信息 (51)。在此极限下，仅具有短程关联的态 (如贝尔网络态) 满足互信息的零律。另一方面，当压缩矩阵也存在长程非零分量时——例如分量随平均几何的距离倒数衰减 [64]——我们可以发现压缩真空的互信息 $S_{AB|C}(|\Gamma,\gamma\rangle)$ 满足有效面积律。

Summary

摘要

We presented recent developments at the interface of loop quantum gravity and quantum information and discussed applications of entanglement measures to quantum geometry. In particular, after describing the Hilbert space of spin-network states together with its interpretation in terms of quantum geometries (Section ”Wave-Functions and Spin Networks”), we introduced the notions of link entanglement, intertwiner entanglement, and boundary spin entanglement (Section ”Geometry as a Network of Entanglement”). We then showed how these notions encode the gluing of quanta of space and their relevance for the reconstruction of a quantum geometry from a network of entanglement structures (Section ”Boundary Spins and Bulk Reconstruction”). Moreover, using information theoretic bounds on the uncertainty of geometric observables and on their correlations (Section ”Entanglement Entropy Bounds on Uncertainties and Correlations”), we showed how the geometric entanglement entropy of spin-network states at fixed spins, treated as a many-body system of quantum polyhedra, allows us to identify a hierarchy of volume-law, area-law, and zero-law states with different scaling laws for correlation functions (Section ”Hierarchy of States: Volume-Law, Area-Law, and Zero-Law States”). In particular, we conjectured area-law states as the corner of the Hilbert space that encodes a semiclassical geometry (Section ”Area-Law States and the Architecture of Spacetime Geometry”) and the geometric entanglement entropy as a probe of semiclassicality in quantum gravity.

我们介绍了圈量子引力与量子信息交叉领域的最新进展，讨论了纠缠度量在量子几何中的应用。具体而言，我们在描述自旋网络态希尔伯特空间及其量子几何诠释 (章节“波函数与自旋网络”) 后，引入了连接纠缠、intertwiners 纠缠与边界自旋纠缠的概念 (章节“作为纠缠网络的几何”)。随后我们说明了这些概念如何编码空间量子的粘合，以及它们对于从纠缠结构网络重构量子几何的意义 (章节“边界自旋与体重构”)。此外，利用信息论对几何观测量不确定性及其关联的界 (章节“不确定性与关联的纠缠熵界”)，我们展示了如何将固定自旋下的自旋网络态几何纠缠熵视为多体量子多面体系统，从而区分出关联函数满足不同标度律的体积律、面积律与零律态层级 (章节“态的层级: 体积律、面积律与零律态”)。具体而言，我们猜想面积律态是希尔伯特空间中编码半经典几何的区域 (章节“面积律态与时空几何的结构”)，并提出几何纠缠熵可作为探测量子引力中半经典性的工具。

The goal of these investigations is two-fold: (i) to determine the fundamental nature of spacetime geometry, by clarifying the role that entanglement plays in gluing spacetime quanta, and (ii) to provide new tools, both conceptual and numerical, for identifying the regime of loop quantum gravity where an effective description in terms of quantum fields on a classical spacetime is valid. These are needed steps in order to extract robust observational predictions from loop quantum gravity. While the results described in this chapter mostly focus on the kinematics, we expect that these techniques can provide the basis for future investigations of the dynamics of loop quantum gravity. In the Hamiltonian framework, squeezed spin networks can provide a variational ansatz for the solution of the Hamiltonian constraint, with the variational parameters encoding directly the entanglement structure of the state. In the covariant framework, bulk reconstruction requires the entanglement structure in the boundary state to match the structure of the spinfoam dynamics.

这些研究有双重目标:(i) 通过厘清纠缠在粘合时空量子过程中发挥的作用，确定时空几何的基本本质；(ii) 提供新的概念与数值工具，确定圈量子引力中可以用经典背景上量子场进行有效描述的适用区间。这些都是从圈量子引力得到可靠观测预言的必要步骤。虽然本章介绍的结果主要聚焦于运动学层面，我们预期这些技术可以为未来研究圈量子引力动力学提供基础。在哈密顿框架中，压缩自旋网络可以为哈密顿约束的解提供变分拟设，变分参数可直接编码态的纠缠结构。在协变框架中，体重构要求边界态的纠缠结构与自旋泡沫动力学结构匹配。

Acknowledgments E.B. acknowledges support from the National Science Foundation, Grant No. PHY-2207851, and from the John Templeton Foundation via the ID 62312 grant, as part of the ”Quantum Information Structure of Spacetime (QISS)” project (qiss.fr).

致谢:E.B. 感谢美国国家科学基金会 (项目号 PHY-2207851) 以及约翰·邓普顿基金会通过 ID 62312 项目提供的支持，本研究是“时空量子信息结构 (QISS)”项目 (qiss.fr) 的一部分。

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